A. M. Baldin, V. I. Gol'danskii, I. L. Rozenthal

An authoritative and comprehensive treatment, of both classical and quantum kinematics of nuclear reactions, for the experimental nuclear physicist.

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PERGAMON PRESS INC.,
122 East 55th Street, New York 22, N.Y.
P.O. Box 47715, Los Angeles, California.

PERGAMON PRESS S.A.R.L.
24 Rue des Ecoles, Paris Ve.

PERGAMON PRESS G.m.b.H.
Kaiserstrasse 75, Frankfurt-am-Main.

A translation of the original volume
"*Kinematika yadernykh reaktsii*"
Moscow, Fizmatgiz, 1959

Library of Congress Card No. 60-53551

Printed in Great Britain by
PERGAMON PRINTING & ART SERVICES LTD.,
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PREFACE TO THE ENGLISH EDITION

We have pleasure in accepting the Publisher's invitation to write a short preface to the English edition of our book.

A number of years have passed since we finished the main part of our work on the manuscript. Many new and interesting papers on nuclear physics and the physics of elementary particles have appeared in this period. Various kinematic relations have been used in these papers, sometimes in new forms. It is natural therefore that we should look at our book with different eyes, 'from the outside' as it were. There are many things with which we are not longer satisfied; we should like to discuss certain topics more fully and treat others differently. Perhaps this is the usual fate of all authors.

We should like to hope, however, that our book will attract the attention of readers outside the Soviet Union and be of some use to them. We shall be grateful to all who send in their comments and criticism and shall try to take them into consideration in future.

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P R E F A C E

The book which is here brought to the reader's attention is entitled 'Kinematics of Nuclear Reactions'. By kinematics an assemblage of relationships based on the Laws of Conservation, which result from the properties of symmetry of space-time in their classical and quantum form, is understood.

The book consists of two parts. With the two introductory chapters at the commencement of the first part, the reader is reminded briefly of some of the basic characteristics of motion at relativistic velocities, and also accounts are given of statements of the utmost importance concerning relativistic transformations. Two extensively used systems of co-ordinates - the laboratory system and a system associated with centre of mass are defined here, and formulae are given for transformation from one of these systems into the other.

Subsequent chapters of the first part are devoted to the classical kinematics of interactions (collisions and decays) involving two, three or more secondary particles. If only two particles are present in the final state, a perfectly definite relationship exists between their directions of motion and also between the direction of motion of a particle and its energy. Functions characteristic of such a relationship are presented in analytical as well as graphical form. Particular cases of non-relativistic interactions and transformations with photon participation are considered separately. If in the final state three or more particles are formed, the relationship between their directions of motion and their energies is not well-defined, and in these cases one is confined to the evaluation of various limiting relationships. Analysis of the angular and energy transformations of multiple processes is derived using Fermi's Statistical Theory. To justify the inclusion of this section in a book on kinematics, it is pointed out that in Fermi's primary concept, the angular and energy distributions of multiple processes are based on the Laws of Conservation of Energy and Momentum.

The second part of the book is devoted to quantum-mechanical analysis of the kinematics of nuclear reactions.

In this part, the scattering matrix (S-matrix), a fundamental concept which is widely used in the interpretation of nuclear interactions, is analysed and its properties are discussed. The S-matrix and Dirac's theory of transformation are used to give an account of the properties of the cross-sections of nuclear reactions which are associated with the Laws of Conservation. The application of Dirac's theory of transformation permits a simple introduction - without the use of group theory - to the various vector addition coefficients used in theories of nuclear reactions (Clebsch-Gordon coefficients, Racah coefficients, Z-coefficients, X-coefficients).

We have allotted a relatively large space to the consideration of time reversal in quantum mechanics (Section 21). This is because the majority of works on the general theory of the S-matrix contains a number of inaccuracies, associated with an incomplete consideration of this problem. These inaccuracies are so prevalent that it is now necessary to give a thorough restatement of the problem, in order to avoid misunderstandings arising from the use of formulae available in the literature.

The specially important case of nuclear reactions which involve photons is discussed separately.

The second section of the book is confined to the discussion of strictly nuclear reactions. Problems of the decay of particles (e.g. problems of correlation as a result of decay) and reactions with polarized particles are not included here (only questions concerning the initial polarization of particles are considered, and the fundamental relationships involved). To give an account of the problems omitted, it would be necessary to enlarge considerably the scope of the mathematical presentations, and this would substantially increase the size and complexity of this book, which is intended primarily for experimentalists.

We did not, by any means, intend writing a handbook of nuclear reactions, but we have none the less included a number of tables, graphs, numerical data and examples.

Thus, in Appendix I, graphs are given for the relationships between angles and energy for certain widely investigated interactions involving light nuclei or elementary particles. In Appendix II, tables of values of the Clebsch-Gordon and Racah coefficients, and also numerical tables of Z, Z_γ and X-coefficients are given.

The first part of the book has been written by V. I. Gol'danskii and I. L. Rozenthal, and the second part by A.M. Baldin.

Mention should be made of the valuable contribution of V. A. Petrun'kin and A. I. Lebedev in the compilation of Appendix II. They collated and worked out the tables of W, Z, X and Z_γ coefficients.

We realise perfectly well that in this book, which is a first attempt to give a systematic treatment of the kinematics of nuclear reactions, there will be many omissions and shortcomings. We would like to thank in advance readers who take the trouble to become acquainted with this book, and who send us any comments arising from reading the book.

In conclusion the authors would like to express their gratitude to V.B. Berestetskii and G.I. Kopylov for having made a number of valuable comments.

A.M. Baldin

V.I. Gol'danskii

I.L. Rozenthal

PART ONE
CLASSICAL THEORY

CHAPTER I

GENERAL PRINCIPLES OF RELATIVISTIC KINEMATICS

Section 1. Constants of Motion. Conservation Laws

As is well-known from classical mechanics, a system of N particles in the case when their spatial structure is neglected (i.e. when the particles are considered as material points), can be described by means of $3N$ differential equations, corresponding to $6N$ constants of motion, i.e. to quantities which are conserved under the changes taking place within the system. The total number of constants of motion, of course, is fixed by the circumstance that for any instant of time the system is defined by the $3N$ co-ordinates and $3N$ momenta of the particles (see for example [1]). Not all of the $6N$ constants of motion are independent*. Let us consider an isolated system, i.e. a system which is not subjected to the action of external forces**. Such a system has ten constants of motion, which correspond to physical quantities, which are unchanged by any arbitrary interaction between the particles of the system during the time of motion. These quantities can be measured, at least in principle, by experiment within the framework of classical mechanics. The 10 constants of motion can be represented, in the following manner: $10 = 4 + 3.2$. The figure 4 corresponds to the Law of Conservation of energy-momentum, which, in relativistic mechanics forms a single four-dimensional vector. These four quantities (the energy and the three components of momentum) are, in the given case, constants of motion. The six remaining constants of motion arise from paired combinations of four axes (three spatial and one

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*A more exact number for the independent constants of motion is equal to $6N = 1$.

**Although such an approach is also somewhat abstract, it nevertheless gives an excellent approximation in all cases of interest to us.

of time). The three quantities obtained by combination of only the spatial axes, correspond to the normal angular momenta, which are constants of motion. The three other quantities, obtained by combination of the time axis and each of the spatial axes, express the rectilinearity and uniformity of the motion of the centre of mass of the system. In Newtonian mechanics the latter statement is a consequence of the Law of Conservation of Momentum.

We shall now dwell upon the important case of the collision of two particles. We shall choose as one of the co-ordinate planes the horizontal plane passing through the trajectory of both particles before collison. In this case, four constants of motion vanish identically (two components of the angular momentum, one component of the momentum, and the velocity of motion of the centre of mass). Thus there remain six, which are essentially of different rights. Actually, as we shall see, the velocity of the system of co-ordinates associated with the centre of mass,is wholly determined by the energies and momenta of the colliding particles, and therefore the remaining constants of motion are not independent.

We have limited our discussion to the realm of classical mechanics. This limitation is reasonable for the cases when one considers energy or momentum. However, for the analysis of the quantities associated with angular momenta, quantum mechanical treatment becomes essential. While energies and momenta of elementary particles can be added up classically, their angular momenta are added in accordance with the principles of quantum mechanics. Since in the first five chapters we shall employ classical concepts, we shall be concerned with energy and momentum. Certain conclusions drawn from an analysis of the conservation of angular momentum will be discussed in the second part of the book.

Section 2. Basic Co-ordinate Systems

Although from the point of view of relativistic mechanics all co-ordinate systems are of equal right, for practical purposes, however, two systems are of particular importance: the laboratory system and the centre of mass system. The laboratory system (L-system) is tied to the earth, as is the observer, consequently all direct observation is in the laboratory system, so it is convenient to use it for reporting experimental results. If we are interested in the process of collision of two particles, then we shall assume that one of them, which we shall denote by the index II, is at rest in the laboratory system, i.e. it has a momentum $p_{II}=0$ (and in the particular case of disintegration of a moving particle I, m_{II} is also equal to 0). In passing, we note that this condition is characteristic of processes in which there are one or two particles in the initial state.

The other important system of co-ordinates is tied to the centre of mass of a system of interacting particles, which in this system is at rest (CM-system). This system is convenient in that in it disintegration and collision processes for two particles have the maximum degree of symmetry. Thus for example if there are no polarization effects, the disintegration of one particle into two others is characterized by a spherically symmetrical distribution of the secondary particles. The existence of polarization makes the symmetry axial. In the collision of two identical particles in the CM-system, in addition to the trivial axis of symmetry coincident with the relative direction of motion of both particles, there is also a plane of symmetry perpendicular to this direction and passing through the point at which the collision occurred.

For the collision of two unequal particles, there is only an axis of symmetry, but it is possible in general to draw certain conclusions concerning the distribution of the particles relative to a reference plane. We shall show that relativistic transformations of quantities from the CM-system into the L-system and vice versa have a particularly simple form compared with transformations into another system (Section 5). We shall mention other special features of the CM-system. Basically its property (in fact the definition of the CM-system) consists in that the total momentum of all interacting particles in this system is equal to zero:

$$\sum_{i=1}^{N} \tilde{p}_i = 0. \tag{2.1}$$

Consequently, the directions of motion of two interacting particles in the CM-system always make an angle of 180° : $\tilde{\vartheta}_1+\tilde{\vartheta}_2=\pi$, i.e. they move directly towards one another before the collision and fly apart in opposite directions after the collision. Because of this, the elements of the solid angles for both interacting particles in the CM-system are always identical (i.e. $|d\cos\tilde{\vartheta}_1|=|d\cos\tilde{\vartheta}_2|$), and their angular distributions are not changed by interchange with the direction of motion of the other particle, which considerably simplifies the interpretation of results. We shall note further that the interaction of the particles is determined by the magnitude of the energies of their relative motion, regardless of the fact that the energy of motion of each of the particles is relative to the observer. Consequently, the energy which can be liberated in nuclear transformation, is given by the total energies of the particles in the CM-system but not in any other. The use of values for the energies of the particles in the CM-system considerably facilitates, in particular, the calculation of the energy thresholds of endothermic nuclear transformations, in which the sum of the masses of the secondary particles exceeds the sum of the masses of the incident particles.

Section 3. Some Formulae of Relativistic Mechanics

We shall consider briefly certain consequences of the theory of relativity (see, for example [2]). In non-relativistic mechanics the fundamental quantities are three-dimensional vectors (e.g. momentum, force etc). The theory of relativity relates space and time in a single four-dimensional continuum, in which the fundamental quantities no longer form three-dimensional but four-dimensional vectors. Thus the length of these vectors will be invariant against rotations of the four-dimensional co-ordinate system. We shall consider in this book only this transformation.

We shall designate the constant velocity of motion of one of the systems of co-ordinates relative to the other by V and we shall assume for simplicity that its direction coincides with the axes x_1 and x_2 of both systems. Let a four-dimensional vector ρ be given in both systems, with components $\rho_{x1}, \rho_{y1}, \rho_{z1}$ and ρ_{t1} (system 1) and $\rho_{x2}, \rho_{y2}, \rho_{z2}, \rho_{t2}$ (system 2). In the case we are discussing, the spatial components of the vector ρ along the axes y and z will not be changed as a result of a transformation from one system to the other, i.e.

$$\rho_{y1} = \rho_{y2}, \quad \rho_{z1} = \rho_{z2}. \tag{3.1}$$

Then, from the condition of invariance of the length of the four-dimensional vector, it follows that

$$\rho_{x1}{}^2 + \rho_{t1}{}^2 = \rho_{x2}{}^2 + \rho_{t2}{}^2. \tag{3.2}$$

The most general linear transformation connecting the co-ordinates $\rho_{x2}, \rho_{t2}, \rho_{x1}$ and ρ_{t1} may be written in the form

$$\rho_{x2} = a\rho_{x1} + b\rho_{t1}, \quad \rho_{t2} = c\rho_{x1} + d\rho_{t1}, \tag{3.3}$$

where a, b, c, d are constants.

Substituting (3.3) in (3.2) and comparing coefficients with respect to identical powers of ρ_{x1} and ρ_{t1}, we obtain

$$a = d, \tag{3.4}$$

$$b = -c, \tag{3.5}$$

$$b = \frac{A}{\sqrt{1+A^2}}, \tag{3.6}$$

$$d = \frac{1}{\sqrt{1+A^2}}, \tag{3.7}$$

where

$$A = \frac{\rho_{x2}}{\rho_{t2}}. \tag{3.8}$$

In the derivation of (3.4) - (3.8) we have used equation (3.3) for $\rho_{x1} = 0$. The quantity A has a simple physical meaning which we shall now consider.

Let us consider two four-dimensional vectors, which we will use frequently: the space-time vector (x, y, z, it) and the energy-momentum vector (p_x, p_y, p_z, iE) *.

Substituting the values of the components of these vectors in (3.8) we obtain

$$A = \frac{x}{it} = -iV, \tag{3.9}$$

where V is the velocity of the first system relative to the second system. Hence,

$$\left.\begin{aligned} \rho_{x2} &= \gamma(\rho_{x1} - iV\rho_{t1}), \\ \rho_{t2} &= \gamma(\rho_{t1} - iV\rho_{x1}) \end{aligned}\right\} \tag{3.10}$$

or

$$\left.\begin{aligned} p_{x2} &= \gamma(p_{x1} + E_1 V), \\ E_2 &= \gamma(E_1 + p_{x1} V). \end{aligned}\right\} \tag{3.11}$$

*Here and henceforth we assume the velocity of light $c = 1$.

where the so-called coefficient of relativistic transformation is

$$\gamma = \frac{1}{\sqrt{1-V^2}}. \qquad (3.12)$$

Let one of the co-ordinate systems be associated with a free particle. Then (3.8) can be rewritten for it in the following form:

$$A = -i\beta = -i\,\frac{p}{E}, \qquad (3.13)$$

where β is the velocity of motion of the particle in the co-ordinate system considered by us, p and E are here, in the following , the momentum total energy of the particle. Consequently

$$\beta = \frac{p}{E}. \qquad (3.14)$$

From the condition of invariance of the square of the modulus of the four-dimensional vector of energy-momentum, we obtain

$$E^2 - p^2 = \mathrm{Inv}. \qquad (3.15)$$

Combining with (3.14) we find

$$E = \frac{\mathrm{Inv}}{\sqrt{1-\beta^2}}. \qquad (3.16)$$

Let us consider the case where $\beta \ll 1$; then

$$E = \mathrm{Inv}\cdot\left(1 + \frac{\beta^2}{2}\right). \qquad (3.17)$$

In order to determine the constant entering in (3.17), we demand that this equation should go over into the correct expression $E = m\frac{\beta^2}{2}$ for the kinetic energy in Newtonian mechanics.

Since, in this case, the energy is accurately defined up to an unessential constant, we have

$$\mathrm{Inv} = m, \qquad (3.18)$$

$$E = \frac{m}{\sqrt{1-\beta^2}}. \qquad (3.19)$$

If the particle is at rest, then it immediately follows that

$$E = m. \qquad (3.20)$$

The circumstance that energy and momentum are components of a single four-dimensional vector, changes the formulation of the Conservation Law as compared with its classical non-relativistic form*. While in classical mechanics the Conservation Laws of energy and momentum emerge as two independent laws, the theory of relativity combines them in the Conservation Law of a four-dimensional vector of energy-momentum. This in turn leads to a most important conclusion concerning the invariance of its absolute value [Equation (3.15)].

Section 4. Relativistic Transformation of Angles and Momenta

In this paragraph we shall consider two problems, the solution of which will be used frequently in future.

1. Let system *1* move along the axis x with a velocity V relative to system *2* . In system *1* the particle moves with a velocity β_1 at an angle ϑ_1 to the axis x . It is required to determine the angle ϑ_2 between the direction of motion of the particle and the x axis in system *2* .

Let the velocity β_1 lie in the plane xy . Assuming that $dx, dy, dz, i\,dt$ form vectors, and taking into account (3.10), it is easy to obtain

$$\beta_{x2} = \frac{\beta_{x1} + V}{1 + \beta_{x1}V}, \quad \beta_{y2} = \frac{1}{\gamma}\,\frac{\beta_{y1}}{1 + \beta_{x1}V}, \quad \beta_{z2} = \beta_{z1} = 0.$$

Since $\mathrm{tg}\,\vartheta_1 = \frac{\beta_{y1}}{\beta_{x1}}$ and $\mathrm{tg}\,\vartheta_2 = \frac{\beta_{y2}}{\beta_{x2}}$ then

$$\mathrm{tg}\,\vartheta_2 = \frac{1}{\gamma}\,\frac{\sin\vartheta_1}{\cos\vartheta_1 + \frac{V}{\beta_1}}. \qquad (4.1)$$

*Strictly speaking, the vector nature of energy-momentum holds good for a system of non-interacting particles (or in cases, considered by us later on, when the particles are very remote from each other (see for example [3]).

Let us consider the important particular case when

$$\frac{V}{\beta_1} \sim 1, \quad 1 + \cos\vartheta_1 \gg 1 - \frac{V}{\beta_1}. \tag{4.2}$$

From (4.1) we obtain

$$\operatorname{tg}\vartheta_2 \approx \frac{1}{\gamma}\operatorname{tg}\frac{\vartheta_1}{2}. \tag{4.3}$$

If $\gamma \gg 1$, then

$$\vartheta_2 \approx \frac{1}{\gamma}\operatorname{tg}\frac{\vartheta_1}{2}. \tag{4.4}$$

We observe also that by using (3.14), equation (4.1) can be written in the form

$$\operatorname{tg}\vartheta_2 = \frac{1}{\gamma}\,\frac{p_1\sin\vartheta_1}{p_1\cos\vartheta_1 + VE_1}. \tag{4.5}$$

2. Let now, in place of the velocity of the particle, its energy E in system 1 be fixed. It is required to determine how the magnitude of E is changed by transformation from system 1 to system 2. The remaining conditions stay as before.

If the particle moves perpendicularly to the direction of $\boldsymbol{V}$; then in consequence of the invariance of the transverse components of the momentum

$$p_1\sin\vartheta_1 = p_2\sin\vartheta_2$$

the energy, on conversion from one system to the other, is not changed. If the particle is moving at an angle to the direction of $\boldsymbol{V}$, then we resolve its momentum into two components: perpendicular to and parallel to $\boldsymbol{V}$. The first component is unchanged by the transformation, the second one is transformed in accordance with (3.11).

$$E_2 = \gamma(E_1 + p_1V\cos\vartheta_1). \tag{4.6}$$

Using (3.15) and (3.18), it is easy to obtain

$$p_2 = \gamma\sqrt{(E_1 + p_1V\cos\vartheta_1)^2 - \frac{m^2}{\gamma^2}}. \tag{4.7}$$

Section 5. Calculation of the γ-coefficient of Relativistic Transformations from the CM-system to the L-system

For the L- and CM-systems, the coefficient of relativistic transformation, in the cases of interest to us, is particularly simply expressed in terms of the most important characteristic of the processes - the total energy of the system.

In the case of disintegration of a single particle I

$$\gamma = \frac{E_I}{m_I}. \tag{5.1}$$

We shall consider the collision of two particles. As already mentioned, it is generally assumed in this case that one of the particles (mass m_{II}) in the L-system is at rest. Then the velocity of this particle in the CM-system coincides with V the velocity of motion of the CM-system relative to the L-system. Hence one can write

$$\gamma = \frac{\widetilde{E}_{II}}{m_{II}}. \tag{5.2}$$

It is sometimes convenient to express the coefficient γ in terms of the energy of the moving particle in the L-system. For this we shall use the equation for the momenta of two interacting particles in the CM-system

$$\widetilde{p}_I = \widetilde{p}_{II}.$$

Hence we obtain

$$\widetilde{E}_{II}^2 = \widetilde{E}_I^2 - m_I^2 + m_{II}^2 \tag{5.3}$$

and

$$\left(\frac{E_I + m_{II}}{\gamma} - m_{II}\gamma\right)^2 = m_{II}^2(\gamma^2 - 1) + m_I^2. \tag{5.4}$$

Solving this equation for γ we find,

$$\gamma = \frac{E_I + m_{II}}{\sqrt{2(E_I + m_{II})m_{II} + m_I^2 - m_{II}^2}}. \tag{5.5}$$

We shall establish a further relationship between the values of the energy of the colliding particles in both systems. Combining (3.11), (5.2) and (5.3) we obtain

$$\widetilde{E}_I = \frac{(E_I + m_{II}) m_{II}}{\sqrt{2(E_I + m_{II}) m_{II} + m_I^2 - m_{II}^2}}. \qquad (5.6)$$

Expressions (5.5) and (5.6) assume a particularly simple form in certain special cases. If $m_I = m_{II}$, then

$$\gamma = \sqrt{\frac{E_I + m_I}{2m_I}}, \qquad (5.7)$$

$$\widetilde{E}_I = \sqrt{\frac{(E_I + m_I) m_I}{2}}. \qquad (5.8)$$

If in addition, $\gamma \gg 1$ (which is equivalent to the conditions that $E_I \gg m_I$), then

$$\gamma \approx \sqrt{\frac{E_I}{2m_I}}, \qquad (5.9)$$

$$\widetilde{E}_I \approx \sqrt{\frac{E_I m_I}{2}}. \qquad (5.10)$$

CHAPTER II

EFFECTIVE CROSS SECTIONS AND THEIR TRANSFORMATION INDUCED BY CHANGE OF THE CO-ORDINATE SYSTEM

Section 6. Integral and Differential Cross-sections

In the study of nuclear interactions there are extremely widespread problems, which may be formulated in the following manner: a given flux j of bombarding particles I penetrates a certain area in which a target particle or nucleus II is at rest; As a result of this, there is some probability $\frac{dw}{dt}$ that in unit time the transformation $I + II \to 1 + 2$ occurs, there is also some probability that when this transformation occurs particle 1 (or 2) acquires a momentum lying within the interval p, $p + dp$, i.e. it emerges within the interval between the angles ϑ, $\vartheta + d\vartheta$ to the direction of motion of the bombarding particle I. It is obvious that the probability of transformation in unit time is proportional to the flux j of the bombarding particles: $\frac{dw}{dt} = \sigma j$. The coefficient of proportionality σ, having the dimension of area (cm^2), is called the coefficient of effective cross-section (or simply the cross-section) of the interaction under consideration. It is customary to distinguish cross-sections by integral and differential. Integral cross-sections characterize the total probability of any transformation without relation to the direction of the emerging particle. We shall denote them by the symbol σ *. Differential cross-sections characterize the probability of the same transformation, but for specific angles of scattering of the secondary particles. Differential cross-sections are customarily denoted by the symbol $d\sigma(\vartheta)$. It is obvious

*In nuclear physics, the unit of cross-section is the "barn", equal to $10^{-24} cm^2$.

that

$$\int_0^\pi d\sigma(\vartheta) = \int_0^\pi \sigma(\vartheta)\, d\cos\vartheta = \sigma. \qquad (6.1)$$

We note that the cross-section, defined in this manner, is relativistically invariant, so long as it is expressed through a three-dimensional flux and area. We shall consider in this book only those interactions in which there are two particles in the initial state, and shall derive only transformations between the L-system, in which one of the primary particles was at rest, and the CM-system. In this case the cross-section is invariant. If in the initial state there are more than two particles, or if conversion is effected to a system of co-ordinates moving at an angle to the direction of the relative motion of the particles, the invariant cross-section should be expressed through a four-dimensional quantity (e.g. a four-dimensional flux).

Section 7. Relativistic Transformation of Angular and Momentum Distributions (Elements of Phase Space)

In the theory of elementary particles and their collision, the following problem is encountered. Let the angular and momentum (or energy) distribution of the particles in one of the systems of co-ordinates be given by the function

$$n_1(p_1, \cos\vartheta_1)\, dp_1 d\cos\vartheta_1\, d\varphi_1\,{}^{*)}$$

whereupon

$$\int\int n_1 dp_1 d\cos\vartheta_1 d\varphi_1 = N,$$

where N is the total number of particles of a specific kind in the system in the final state.

It is required to determine the corresponding function

$$n_2(p_2, \cos\vartheta_2)\, dp_2 d\cos\vartheta_2\, d\varphi_2$$

in the other system of co-ordinates. It is obvious that the problem is completely equivalent to the solution of the following problem: let the differential cross-section be given in the form

$$d\sigma = \frac{\sigma}{N} n_1(p_1, \cos\vartheta_1)\, dp_1 d\cos\vartheta_1\, d\varphi_1; \qquad (7.1)$$

It is necessary to find an expression for the transformed differential cross-section, which in this case is determined (with an accuracy up to an approximately constant factor) by the nature of the transformation of the function n_1. The transformation from n_1 to n_2 is the same as the transformation from the system of the variables p_1, ϑ_1 to p_2, ϑ_2, which is effected, as already mentioned, by means of the Jacobian

$$J = \frac{\partial p_1}{\partial p_2}\frac{\partial \cos\vartheta_1}{\partial \cos\vartheta_2} - \frac{\partial p_1}{\partial \cos\vartheta_2}\frac{\partial \cos\vartheta_1}{\partial p_2}. \qquad (7.2)$$

Thus, as a result of conversion from one system of co-ordinates to the other, the angular and momentum distributions are transformed in the following manner:

$$\begin{aligned} n_1(p_1, \cos\vartheta_1)\, dp_1 d\cos\vartheta_1 d\varphi_1 &= n_1\left[p_1(p_2, \cos\vartheta_2); \cos\vartheta_1(p_2, \cos\vartheta_2)\right]\cdot \\ J dp_2 d\cos\vartheta_2\, d\varphi_2 &= n_2(p_2, \cos\vartheta_2)\, dp_2 d\cos\vartheta_2\, d\varphi_2. \end{aligned} \qquad (7.3)$$

Using (4.5), (4.7) and the equation $d\varphi_1 = d\varphi_2$, it is possible to obtain

$$J = \gamma \frac{p_2^2}{p_1^2 E_2}(E_2 - p_2 V\cos\vartheta_2) = \frac{p_2^2 (E_2 - p_2 V\cos\vartheta_2)}{\gamma E_2\left[(E_2 - p_2 V\cos\vartheta_2)^2 - \frac{m^2}{\gamma^2}\right]}. \qquad (7.4)$$

A particularly simple Jacobian is expressed in the energy representation when the quantities E and $\cos\vartheta$ are used. For evaluation of the Jacobian J in this case, we make use of the invariant quantity $\frac{dp_x dp_y dp_z}{E}$ * and we express the element of the phase volume in the form

$$dp_x dp_y dp_z = p^2 dp\, d\cos\vartheta\, d\varphi. \qquad (7.5)$$

Since the number of particles in the specified element of

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*The introduction of n_1 in such a form is convenient in that the solid angle dependence is given immediately, the element of solid angle being equal to $\sin\vartheta\, d\vartheta$.

*Proof of the invariance is obtained readily by simple calculation (See for example [1, 2]).

the phase volume is invariant, then

$$\frac{n_1(E_1, \cos\vartheta_1)\, p_1^2 dp_1 d\cos\vartheta_1}{E_1} = \frac{n_1\left[E_1(E_2, \cos\vartheta_2), \cos\vartheta_1(E_2, \cos\vartheta_2)\right] p_2^2 dp_2 d\cos\vartheta_2}{E_2} \qquad (7.6)$$

from whence it follows

$$J = \frac{p_2^2 E_1}{p_1^2 E_2}. \qquad (7.7)$$

In the system of co-ordinates (p_1, ϑ_1), the area which encloses the values p_1, ϑ_1 for any given process, is defined by the curves

$$p_{1\,max} = \Phi_1(\vartheta_1) \qquad (7.8)$$

and

$$p_{1\,min} = \Phi_2(\vartheta_1), \qquad (7.9)$$

where Φ_1, Φ_2 are certain functions defined by physical processes and Conservation Laws. Thus, for reactions in which there are three or more particles in the final state, the momenta of the secondaries can assume any value from 0 to p_{max}, determined by the magnitude of the masses of the secondary particles. For reactions involving only two secondary particles, the momenta in the CM-system assume definite values, depending only on the values of $\widetilde{E}, m_1, m_2$ (For more detail concerning this, see the following section).

Substituting in (7.8) and (7.9) the values of the quantities ϑ_1 and p_1, determined in accordance with (4.1) and (4.7), we obtain the following equations for the curves which define the limits of variation of the co-ordinates p_2, ϑ_2:

$$\gamma\sqrt{(E_2 - p_2 V\cos\vartheta_2)^2 - \frac{m^2}{\gamma^2}} = \Phi_1\left[\operatorname{arctg}\frac{1}{\gamma}\left(\frac{p_2\sin\vartheta_2}{p_2\cos\vartheta_2 - VE_2}\right)\right], \qquad (7.10)$$

$$\gamma\sqrt{(E_2 - p_2 V\cos\vartheta_2)^2 - \frac{m^2}{\gamma^2}} = \Phi_2\left[\operatorname{arctg}\frac{1}{\gamma}\left(\frac{p_2\sin\vartheta_2}{p_2\cos\vartheta_2 - VE_2}\right)\right]. \qquad (7.11)$$

Equations (7.10) and (7.11) are considerably simplified in practice. Thus for example, for disintegration $\Phi_1 = p_{1\,max} = \text{const}$; in the case of disintegration into two particles $\Phi_1 = \Phi_2$; for disintegration into three particles $\Phi_2 = 0$. In these cases the curve corresponding to (7.10) is described by an equation of the second order. Examination shows that this curve is at all times an ellipse. Thus, if in the co-ordinate system (p_1, ϑ_1) the curve enclosing a certain area is a circle, then transformation into a system of co-ordinates (p_2, ϑ_2) converts it into an ellipse*.

Solving equation (7.10) with respect to p_2 for the case $\Phi_1 = p_{1\,max}$, we obtain

$$p_2 = \frac{E_1 V\cos\vartheta_2 \pm \sqrt{m^2\gamma^2 V^2\cos^2\vartheta_2 - m^2\gamma^2 + E_1^2}}{\gamma(1 - V^2\cos^2\vartheta_2)}. \qquad (7.12)$$

The sign in front of the radical in formula (7.12) is chosen in the following manner:

1) If the expression under the radical is always positive, then the sign + should be chosen. Actually, in the case considered, for a change of momentum from 0 to $p_{1\,max}$, the sign before the radical is unchanged, since the function should be constant. Hence in every momentum interval the sign should be either + or -. But since there are at all times positive values of p_2 (e.g. for $\vartheta_2 = 0$), then in the case considered the sign + should be chosen for every interval $(0 - p_{1\,max})$. Thus, the first criterion for choice of the sign is the positive definiteness of the expression

$$D(\vartheta_2) = m^2\gamma^2 V^2\cos^2\vartheta_2 - m^2\gamma^2 + E_1^2. \qquad (7.13)$$

It is easy to show that $D(\vartheta_2) > 0$ if $\beta_1 > V$, i.e. the velocity of the particle in system *1* is greater than the velocity V of the transformed system relative to system *2*. Consequently, if the velocity of the particle $\beta_1 > V$, then at all times the sign + is chosen.

2) If $\beta_1 < V$, then for a certain value $\vartheta_2 = \vartheta_{2\,max}$, $D(\vartheta_2) = 0$, and it is necessary to take into account both signs. For

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*Examination of the ellipse for the case of elastic collision has been carried out in [2]; general cases are analysed by Bleton [4]. The general principles of construction of such ellipses and a number of examples will be discussed in Section 10.

$\vartheta_2 > \vartheta_{2\,max}$ * we have $D(\vartheta_2) < 0$ (the angle $2\vartheta_{2\,max}$ includes the ellipse given by equation (7.10)).

From the condition that $D(\vartheta_{2\,max}) = 0$, we obtain

$$\sin\vartheta_{2\,max} = \sqrt{\frac{1 - V^2}{\frac{V^2}{\beta_1^2} - V^2}}. \qquad (7.14)$$

Figure 1 represents a diagram of the Lorentz transformation of the circle $p_1 = \text{const}$ for the case $\beta_1 > V$. Between the radius-vectors of the circle and the ellipse there is a simple relationship. The ellipse has been computed according to formula (7.12) for the following values of the parameters: $m = 1$; $E_1 = 3$; $V = 0.9$.

It is convenient to use the method of transformation described to obtain the angular and energy functions in the L-system (see Section 13).

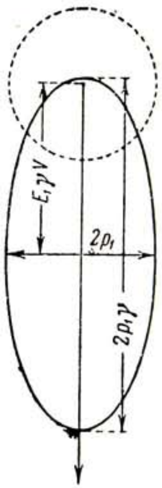


Fig. 1

Transformation $p_1,\vartheta_1 \rightarrow p_2,\vartheta_2$: $\beta_1 > V$; $E_1 = 3$; $V = 0.9$.

*The limit on the angle θ_2 for the condition $\beta_1 < V$ reflects the fact that in this case the particles in the L-system only move in the forward direction.

Another form of analysis has been investigated by Bradt, Kaplon and Peters [5], using a compound system of co-ordinates (p_1, ϑ_2).

In order to calculate the Jacobian in this case, we use formula (4.5) from which it follows that

$$\cos\vartheta_1 = \frac{-E_1 V \gamma^2 \operatorname{tg}^2\vartheta_2 \pm \sqrt{p_1^2 + [p_1^2 - V^2 E_1^2]\gamma^2 \operatorname{tg}^2\vartheta_2}}{p_1[\gamma^2 \operatorname{tg}^2\vartheta_2 + 1]}. \qquad (7.15)$$

The sign in formula (7.15) is chosen by considerations similar to those presented earlier:

1. If the expression under the radical is always positive, then the positive sign (+) should be chosen for every interval of the angle ϑ_2 (from 0 to π).

The criterion for choice of the sign is the positive definiteness of the expression

$$f(\vartheta_2) = p_1^2 + \gamma^2 \operatorname{tg}^2\vartheta_2 [p_1^2 - V^2 E_1^2], \qquad (7.16)$$

which also occurs for the condition $\beta_1 > V$.

2. If $\beta_1 < V$, the maximum permissible angle $\vartheta_{2\,max}$ is at all times π. From continuity considerations it follows that within the intervals $0 < \vartheta_1 < \vartheta_{1\,max}$ and $\vartheta_{1\,max} < \vartheta_1 < \pi$ ($\vartheta_{1\,max}$ is the angle corresponding to $\vartheta_{2\,max}$), it is necessary to choose a constant sign before the radical. Since $\cos\vartheta > 0$ for the value $\vartheta = 0$, then within every interval for $0 < \vartheta_1 > \vartheta_{1\,max}$ the sign + should be chosen; by analagy, within the interval $\vartheta_{1\,max} < \vartheta_1 < \pi$ the sign - is chosen.

From the condition that $f[\vartheta_2(\vartheta_1)] = 0$, one can write

$$\cos\vartheta_{1\,max} = -\frac{\beta_1}{V}. \qquad (7.17)$$

In system 2 the angle $\vartheta_{2\,max}$ is determined by formula (7.14). Thus, if $\beta_1 < V$ and $\vartheta_1 < \vartheta_{1\,max}$, the sign + is chosen, and for $\vartheta_1 > \vartheta_{1\,max}$ the sign - is chosen.

From (7.15) it is possible to calculate $J_1 = \left.\frac{\partial\cos\vartheta_1}{\partial\cos\vartheta_2}\right|_{p_1 = \text{const}}$.

$$J_1 = \frac{\gamma^2\left[VE_1 \pm \sqrt{p_1^2 + \gamma^2 \operatorname{tg}^2\vartheta_2 (p_1^2 - V^2 E_1^2)}\right]^2}{p_1 \cos^3\vartheta_2 (\gamma^2 \operatorname{tg}^2\vartheta_2 + 1)^2 \sqrt{p_1^2 + \gamma^2 \operatorname{tg}^2\vartheta_2 (p_1^2 - V^2 E_1^2)}} \qquad (7.18)$$

the sign + is chosen if $\beta_1 > V$, or if $\beta_1 < V$ and $0 < \vartheta_1 < \vartheta_{1max}$; the sign is - if $\beta_1 < V$ and $\vartheta_{1\,max} < \vartheta_1 < \pi$.

Figure 2a represents the transformation of the area bounded, in system *1*, by the equations $p_{1max} = \text{const}$; $p_{1min} = 0$ for the case $\beta_1 > V$.

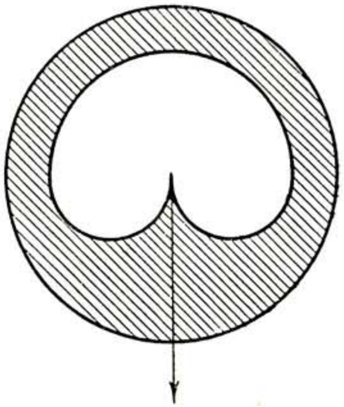


Fig. 2a

Transformation $p_1, \vartheta_1 \to p_1, \vartheta_2$: $\beta_1 > V$; $E_1 = 3$, $V = 0.9$

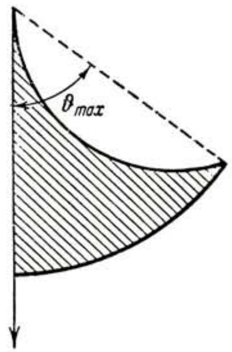


Fig. 2b

Transformation $p_1, \vartheta_1 \to p_1, \vartheta_2$: $\beta_1 < V$; $E_1 = 2$; $V = 0.9$

Naturally, since the momentum is not transformed, then the circle $p_{1max} = \text{const}$. is not changed. For small momenta however, in accordance with formula (7.14) prohibited areas appear. The cross-hatched area corresponds to the permissible values of the momenta and angles in system *2* . Figure 2b shows the transformation of the same area for the condition $\beta_1 < V$. The scale and magnitude of the parameters in both diagrams are identical.

In this case, if the momentum in system *1* has a strictly defined value (e.g. as a result of disintegration into two particles) then for $\beta_1 < V$ the circle $p_{1max} = \text{const}$. is transformed into an arc.

In isolated cases, in order to obtain the momentum distribution, it is convenient to use the system (p_2, ϑ_1) . Thus, in order to obtain the Jacobian J_2 corresponding to transformation from the system (p_1, ϑ_1) , into system (p_2, ϑ_1) , it is

necessary to evaluate the explicit expression for $p_1(p_2, \vartheta_1)$.

Using formula (7.12) we obtain

$$J_2 = \left.\frac{\partial p_1}{\partial p_2}\right|_{\vartheta_1 = \text{const}} = \frac{p_2}{\gamma(1 - V^2\cos^2\vartheta_1)}\left[\frac{1}{\sqrt{p_2^2 - \gamma^2 V^2 m^2 \sin^2\vartheta_1}} - \frac{V\cos\vartheta_1}{\sqrt{p_2^2 + m^2}}\right]. \tag{7.19}$$

The limits of the variable quantities $p_2(\vartheta_1)$ are easily determined from the equations

$$p_{2\,min} = \gamma\sqrt{(E_{1\,min} + p_{1\,min} V\cos\vartheta_1)^2 - \frac{m^2}{\gamma^2}}, \tag{7.20}$$

$$p_{2\,max} = \gamma\sqrt{(E_{1\,max} + p_{1\,max} V\cos\vartheta_1)^2 - \frac{m^2}{\gamma^2}}. \tag{7.21}$$

The values of $p_{2\,min}$ and $p_{2\,max}$ take on a particularly simple form in the case which is of interest to us, when $p_{1\,min} = 0$; $p_{1\,max} = \text{const}$.

The transformation of the area bounded by the circle resulting from the conversion $(p_1, \vartheta_1) \to (p_2, \vartheta_1)$ is presented diagrammatically in Figure 3.

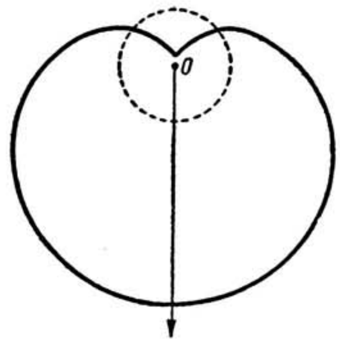


Fig. 3

Transformation $p_1, \vartheta_1 \to p_2, \vartheta_1$: $E_1 = 3$; $V = 0.9$

It is necessary to emphasise, that in the extreme relativistic case, the basic presentations of the above formulae for evaluation of the Jacobians are substantially simplified. If $p_2 \gg m$, $\gamma \gg 1$, then

$$J \approx \frac{p_2}{\gamma(E_2 - p_2 V \cos\vartheta_2)}. \qquad (7.22)$$

For the complementary conditions

$$1 - \cos\vartheta_2 \gg \frac{m^2}{p_2^2},$$

$$J \approx \frac{1}{\gamma(1 - V\cos\vartheta_2)}. \qquad (7.23)$$

If $\gamma \gg 1$ and $\vartheta_2 \ll 1$, then

$$\cos\vartheta_1 \approx \frac{-V\gamma^2 \operatorname{tg}^2\vartheta_2 \pm 1}{\gamma^2 \operatorname{tg}^2\vartheta_2 + 1} \qquad (7.24)$$

and

$$J_1 \approx \frac{\pm\gamma^2 (V \pm 1)^2}{\cos^3\vartheta_2 (\gamma^2 \operatorname{tg}^2\vartheta_2 + 1)^2}. \qquad (7.25)$$

If $\vartheta_2 \ll \frac{1}{\gamma}$, is complementary, then

$$\cos\vartheta_1 \approx 1 - 2\gamma^2\vartheta_2^2 \qquad (7.26)$$

and

$$J_1 \approx 4\gamma^2(1 - \gamma^2\vartheta_2^2). \qquad (7.27)$$

If $p_2 \gg \gamma m$, then

$$J_2 \approx \frac{1}{\gamma(1 + V\cos\vartheta_1)}. \qquad (7.28)$$

In conclusion of this chapter, it should be noted that all the relationships deduced in it are applicable for transformation of angles and momenta for any two systems of coordinates. In the case when conversion from the CM-system to the L-system is considered, however, the formulae are considerably simplified, since in this case the transformation coefficient γ can be expressed in terms of the most important characteristic of the process - the energy of the primary particle (See Section 5).

CHAPTER III

KINEMATICS OF INTERACTIONS INVOLVING TWO SECONDARY PARTICLES

Section 8. Interaction in the General Relativistic Case

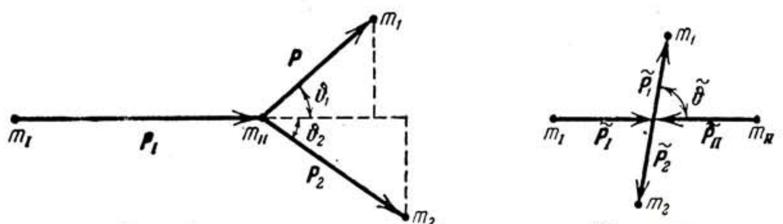
The most prevalent reactions in laboratory practice are nuclear interactions involving two secondary particles. Such interactions include elastic and inelastic scattering, and the vast majority of nuclear reactions in the range of energies up to several hundred MeV. A number of transformations involving two elementary particles - mesons, hyperons, nucleons and antinucleons have been studied in recent years - (e.g. $K^- + p \to \Sigma^+ + \pi^-$, $p + \tilde{p} \to n + \tilde{n}$ etc.), brought about by cosmic rays and ultra-high energy accelerators.

Nuclear interactions involving three secondary particles include, at low energies, the creation of the pair e^-, e^+ and β-decay (the third particle participating in each process is a recoil nucleus). At high energies, interactions with formation of a large number of particles become more probable. As is well-known, at ultra-high energies nuclear processes are observed in which tens of elementary particles are sometimes formed.

The problem of the kinematic analysis of nuclear interactions from the laws of conservation of energy and momentum in the first place, concerns the determination (for given masses and energies of the interacting particles) of the relationship between the directions of motion of the various reaction products and between the direction of motion and energy of each one of them. A comparison of the data, calculated and found by experiment, concerning the relationship between the directions of motion and the energies of the products of nuclear interactions, assists their accurate identification, thereby assisting in making a correct choice between the various possible mechanisms of interaction.

Frequently, another problem connected with the kinematics of nuclear reactions emerges - the problem of conversion of the directions of motion, energies of the particles and the effective cross-sections of the processes from the L-system into the CM-system, for which the theoretical formulae for the distributions are usually derived. Sometimes the reverse problem is set - conversion from the CM-system into the L-system. Both these problems can be solved completely only in the case where two secondary particles are involved in the process under consideration. With the participation of three or a larger number of particles, it is only possible, as indicated below, to obtain certain limiting relationships associated with various supplementary assumptions. These limiting cases are, in point of fact, different variants of the substitution of a one-act formation of many particles by several acts, in each one of which up to two particles is formed.

Thus, nuclear interactions involving two secondary particles, to which this chapter is devoted, are not only the most common and widespread but are applied also in calculations on processes of other types.



L-system CM-system

Fig. 4

The reaction $\mathrm{I}+\mathrm{II}\to 1+2$ in the L- and CM-systems

We shall now consider a nuclear interaction of the type: $\mathrm{I}+\mathrm{II}\to 1+2$ (See Figure 4)*.

*Here, and henceforth the indices 1 and 2 relate to the end products of the reactions, and not to the systems of coordinates, as previously.

As usual we shall assume that in the L-system only one of the interacting particles is moving (I), and the target particle (II) is at rest. Then the relativistic expressions for the Laws of conservation of energy and momentum will take the form:

$$E_{\mathrm{I}}+m_{\mathrm{II}}=E_1+E_2=E_{\mathrm{T}}, \qquad (8.1)$$

$$p_{\mathrm{I}}=p_1\cos\vartheta_1+p_2\cos\vartheta_2, \qquad (8.2)$$

$$0=p_1\sin\vartheta_1-p_2\sin\vartheta_2, \qquad (8.3)$$

where E_{T} is the total energy of the particles. Hence we obtain the following relationship between the directions of motion and the momenta of particles 1 and 2;

$$E_1=\frac{A_1E_{\mathrm{T}}\pm p_{\mathrm{I}}\cos\vartheta_1\sqrt{A_1^2-4m_1^2\left(E_{\mathrm{T}}^2-p_{\mathrm{I}}^2\cos^2\vartheta_1\right)}}{2\left(E_{\mathrm{T}}^2-p_{\mathrm{I}}^2\cos^2\vartheta_1\right)}, \qquad (8.4)$$

$$E_2=\frac{A_2E_{\mathrm{T}}\pm p_{\mathrm{I}}\cos\vartheta_2\sqrt{A_2^2-4m_2^2\left(E_{\mathrm{T}}^2-p_{\mathrm{I}}^2\cos^2\vartheta_2\right)}}{2\left(E_{\mathrm{T}}^2-p_{\mathrm{I}}^2\cos^2\vartheta_2\right)}, \qquad (8.4')$$

where

$$A_1=E_{\mathrm{T}}^2-p_{\mathrm{I}}^2+m_1^2-m_2^2=M_{\mathrm{N}}^2+2m_{\mathrm{II}}W_{\mathrm{I}}+m_1^2-m_2^2, \qquad (8.5)$$

$$A_2=E_{\mathrm{T}}^2-p_{\mathrm{I}}^2+m_2^2-m_1^2=M_{\mathrm{N}}^2+2m_{\mathrm{II}}W_{\mathrm{I}}+m_2^2-m_1^2, \qquad (8.5')$$

W_{I} is the kinetic energy of particle I.

The total rest mass of the system in the initial state is

$$M_{\mathrm{N}}=m_{\mathrm{I}}+m_{\mathrm{II}}. \qquad (8.6)$$

The requirement for positiveness of the term under the root sign in expressions (8.4) and (8.4') lead, as already observed in Section 7, to the conditions limiting the directions of motion of particles 1 and 2 :

$$\sin^2\vartheta_1\leqslant\frac{A_1^2-4m_1^2\left(M_{\mathrm{N}}^2+2m_{\mathrm{II}}W_{\mathrm{I}}\right)}{4m_1^2p_{\mathrm{I}}^2}, \qquad (8.7)$$

$$\sin^2\vartheta_2\leqslant\frac{A_2^2-4m_2^2\left(M_{\mathrm{N}}^2+2m_{\mathrm{II}}W_{\mathrm{I}}\right)}{4m_2^2p_{\mathrm{I}}^2}. \qquad (8.8)$$

It is obvious that assuming ϑ_1 and ϑ_2 in (8.7) and (8.8) are equal to zero, we should obtain the value of $W_{I\,\text{Thresh.}}$, corresponding to the threshold of the reaction being studied, which is reached when the energy available in the CM-system is equal to some value, Q, given by

$$Q = M_N - (m_1 + m_2) = M_N - M_\kappa \tag{8.9}$$

(M_κ is the rest mass of the system in the final state).

Proceeding from the relationship

$$A_1^2 = 4m_1^2 (M_N^2 + 2m_{II} W_{I\,\text{Thresh.}}) \tag{8.10}$$

or

$$A_2^2 = 4m_2^2 (M_N^2 + 2m_{II} W_{I\,\text{Thresh.}}) \tag{8.10'}$$

we obtain

$$M_N^2 + 2m_{II} W_{I\,\text{Thresh.}} = M_\kappa^2, \tag{8.11}$$

whence

$$2m_{II} W_{I\,\text{Thresh.}} = M_\kappa^2 - M_N^2 = |Q|(|Q| + 2M_N) \tag{8.12}$$

or

$$W_{I\,\text{Thresh.}} = |Q| \left\{ \frac{M_N + \frac{1}{2}|Q|}{m_{II}} \right\}. \tag{8.13}$$

Thus, we have established two fundamental kinematic functions $E_1 = f(\vartheta_1)$ and $E_2 = f(\vartheta_2)$. It is obvious that a third fundamental function $\vartheta_2 = f(\vartheta_1)$ can be obtained from them by applying the Law of Conservation of energy. In order to obtain this function in an analytical form it is easier, however, to use the formula for transformation from the L- to the CM-system. The velocity of motion V of the CM-system relative to the L-system is given by

$$V = \frac{p_I + p_{II}}{E_I + E_{II}}, \tag{8.14}$$

and in the case when particle II is at rest

$$V = \frac{p_I}{E_I + m_{II}} = \frac{p_I}{E_T}. \tag{8.14'}$$

Hence it is obvious that the velocities of particles I and II in the CM-system are equal to

$$\tilde{\beta}_{II} = V = \frac{\sqrt{E_I^2 - m_I^2}}{E_I + m_{II}} = \frac{\sqrt{W_I(W_I + 2m_I)}}{W_I + M_N}, \tag{8.15}$$

$$\tilde{\beta}_I = \frac{\beta_I - V}{1 - \beta_I V} = \frac{m_{II}\sqrt{E_I^2 - m_I^2}}{m_I^2 + m_{II} E_I}. \tag{8.15'}$$

The momenta of the approaching particles I and II in the CM-system will be

$$\tilde{p}_I = \tilde{p}_{II} = \tilde{p}_N = m_{II} \sqrt{\frac{E_I^2 - m_I^2}{m_I^2 + m_{II}(2E_I + m_{II})}}, \tag{8.16}$$

and their energies

$$\tilde{E}_I = \frac{m_I^2 + m_{II} E_I}{\sqrt{m_I^2 + m_{II}(2E_I + m_{II})}}, \quad \tilde{E}_{II} = \frac{m_{II}(E_I + m_{II})}{\sqrt{m_I^2 + m_{II}(2E_I + m_{II})}}. \tag{8.17}$$

Hence we obtain an expression for the total energy of the two approaching particles in the CM-system:

$$\tilde{E}_T = \tilde{E}_I + \tilde{E}_{II} = \sqrt{m_I^2 + m_{II}(2E_I + m_{II})} = \sqrt{M_N^2 + 2m_{II} W_I}, \tag{8.18}$$

i.e.

$$\tilde{E}_T^2 = E_T^2 - p_I^2 = p_I^2 \frac{1 - V^2}{V^2} = E_T^2 (1 - V^2). \tag{8.19}$$

Equation (8.19) is a particular case of (3.15). Actually, from the definition of the CM-system - that the total momentum $\tilde{p}_T = 0$, it follows from (3.15) that

$$E_T^2 - p_T^2 = \text{Inv} = \tilde{E}_T^2. \tag{8.20}$$

From equation (8.18), characterizing the total energy in the CM-system, we can obtain an expression for the threshold of the reaction:

$$\tilde{E}_{T,\text{Thresh.}} = \sqrt{M_N^2 + 2m_{II} W_{I\,\text{Thresh.}}} = M_\kappa, \tag{8.21}$$

from which follows the relationship (8.13) above.

From the conditions

$$\widetilde{E}_1+\widetilde{E}_2=\widetilde{E}_T \tag{8.22}$$

and

$$\widetilde{p}_1=\sqrt{\widetilde{E}_1^2-m_1^2}=\widetilde{p}_2=\sqrt{\widetilde{E}_2^2-m_2^2} \tag{8.23}$$

we can easily obtain an expression for the momenta and energies of the two secondary particles in the CM-system:

$$\widetilde{p}_1=\widetilde{p}_2=\widetilde{p}_K=\frac{\sqrt{A_{1(2)}^2-4\widetilde{E}_T^2m_{1(2)}^2}}{2\widetilde{E}_T}=\frac{\sqrt{[\widetilde{E}_T^2-(m_1+m_2)^2][\widetilde{E}_T^2-(m_1-m_2)^2]}}{2\widetilde{E}_T}, \tag{8.24}$$

$$\widetilde{E}_{1(2)}=\frac{A_{1(2)}}{2\widetilde{E}_T}=\frac{\widetilde{E}_T^2+m_{1(2)}^2-m_{2(1)}^2}{2\widetilde{E}_T}. \tag{8.25}$$

We shall now proceed to establish the relationship between the directions of motion in the L- and CM-systems. We shall take as the direction of motion $\widetilde{\vartheta}$ in the CM-system, the angle $\widetilde{\vartheta}_1$ between the direction of the vectors $\widetilde{p}_1$ and $\widetilde{p}_1$. Hence, it is obvious that the angle between the directions of the vectors $\widetilde{p}_1$ and $\widetilde{p}_2$ will be equal to $\widetilde{\vartheta}_2=\pi-\widetilde{\vartheta}_1$. In view of this simple relationship between the angles $\widetilde{\vartheta}_1$ and $\widetilde{\vartheta}_2$, we shall derive all the basic formulae for a single angle $\widetilde{\vartheta}_1=\widetilde{\vartheta}$, i.e. for only one of the reaction products - for particle 1. If it is required to obtain the corresponding formulae for particle 2, it is only necessary to substitute everywhere the index 1 by the index 2, and $\cos\widetilde{\vartheta}$ by $-\cos\widetilde{\vartheta}$. In those circumstances when the formulae are identical for particles 1 and 2, we shall write a generalized index n.

From the relationships:

$$\left.\begin{aligned} p_1\cos\vartheta_1&=\frac{\widetilde{p}_K\cos\widetilde{\vartheta}+V\widetilde{E}_1}{\sqrt{1-V^2}},\\ p_1\sin\vartheta_1&=\widetilde{p}_K\sin\widetilde{\vartheta},\end{aligned}\right\} \tag{8.26}$$

illustrated for the case under consideration in Figure 5, it follows that

$$\operatorname{ctg}\vartheta_1=\frac{\cos\widetilde{\vartheta}+\rho_1}{\sqrt{1-V^2}\sin\widetilde{\vartheta}}, \tag{8.27}$$

where

$$\rho_n=\frac{V}{\widetilde{\beta}_n}=\frac{p_I\widetilde{E}_n}{\widetilde{p}_nE_T}=\frac{\sqrt{E_I^2-m_I^2}}{E_I+m_{II}}\,\frac{A_n}{\sqrt{A_n^2-4\widetilde{E}_T^2m_n^2}}. \tag{8.28}$$

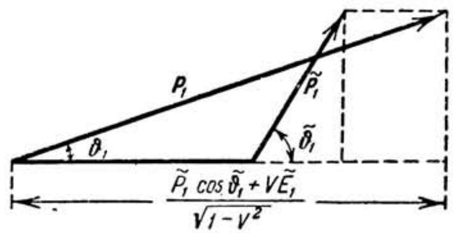


Fig. 5

Diagram of the relationship between the L- and CM-systems (Relativistic case) - addition of velocities

Using the equation

$$\frac{1}{\sqrt{1-V^2}}=\gamma=\frac{E_T}{\widetilde{E}_T}=\frac{E_I+m_{II}}{\sqrt{m_I^2+m_{II}(2E_I+m_{II})}} \tag{8.29}$$

from (8.27) we obtain the inverse function $\widetilde{\vartheta}$ from ϑ_1 in the form

$$\cos\widetilde{\vartheta}=\frac{-\gamma^2\rho_1\pm\operatorname{ctg}\vartheta_1\sqrt{\gamma^2(1-\rho_1^2)+\operatorname{ctg}^2\vartheta_1}}{\gamma^2+\operatorname{ctg}^2\vartheta_1} \tag{8.30}$$

or

$$\cos\widetilde{\vartheta}=\frac{-\rho_1\sin^2\vartheta_1\pm(1-V^2)\cos\vartheta_1\sqrt{1-\frac{\rho_1^2-V^2}{1-V^2}\sin^2\vartheta_1}}{1-V^2\cos^2\vartheta_1}, \tag{8.30'}$$

and also

$$\sin\widetilde{\vartheta}=\frac{\gamma\left[\rho_1\operatorname{ctg}\vartheta_1\pm\sqrt{\gamma^2(1-\rho_1^2)+\operatorname{ctg}^2\vartheta_1}\right]}{\gamma^2+\operatorname{ctg}^2\vartheta_1} \tag{8.31}$$

or

$$\sin\widetilde{\vartheta}=\frac{\sqrt{1-V^2}\sin\vartheta_1\left[\rho_1\cos\vartheta_1\pm\sqrt{1-\frac{\rho_1^2-V^2}{1-V^2}\sin^2\vartheta_1}\right]}{1-V^2\cos^2\vartheta_1}. \tag{8.31'}$$

The formulae relating the angle $\tilde{\vartheta}$ with the angle ϑ_2 are obtained in a precisely similar manner. The condition for the limiting angles ϑ_1 and ϑ_2 is determined by equation (7.14). For $\rho_n \geqslant 1$, to the given angle ϑ_n in the L-system, there corresponds two values of the angle $\tilde{\vartheta}$ in the CM-system, and only for $\vartheta_n = \vartheta_{n\,\max}$ is the solution for $\tilde{\vartheta}$ single-valued, viz.

$$\cos\tilde{\vartheta}_0 = -\frac{1}{\rho_n}. \qquad (8.32)$$

For $\rho_n \geqslant 1$, the term under the root sign in equations (8.30) and (8.31) are positive for any ϑ_n and it follows that the sign before the root will always be positive (+). Hence, to every value of ϑ_n there corresponds only one value of $\tilde{\vartheta}$.

Using (8.26) and (8.30'), we obtain the following expression for the dependence of the momentum of the secondary particle on its direction of motion:

$$p_n(\vartheta_n) = \frac{\tilde{p}_K\sqrt{1-V^2}}{1-V^2\cos^2\vartheta_n}\left\{\rho_n\cos\vartheta_n \pm \sqrt{1-\frac{\rho_n^2-V^2}{1-V^2}\sin^2\vartheta_n}\right\}. \qquad (8.33)$$

Eliminating from (8.27) the direction of motion of particle *1* in the CM-system, $\tilde{\vartheta}$, we find the following expression for the relationship between the angles ϑ_1 and ϑ_2 in the L-system:

$$\operatorname{ctg}\vartheta_2 = \frac{-(1+\rho_1\rho_2)\operatorname{ctg}\vartheta_1 \pm (\rho_1+\rho_2)\sqrt{\gamma^2(1-\rho_1^2)+\operatorname{ctg}^2\vartheta_1}}{1-\rho_1^2}. \qquad (8.34)$$

Also, for $\rho_{1(2)} > 1$, we shall encounter an ambiguous dependence of $\vartheta_{1(2)}$, on the angle of the other particle $\vartheta_{2(1)}$. For $\vartheta_1 = \vartheta_2 = \vartheta$, (8.34) becomes

$$\operatorname{ctg}^2\vartheta = \frac{\gamma^2(\rho_1+\rho_2)^2\left[1-\rho_{1(2)}^2\right]}{\left[\left(1-\rho_{1(2)}^2\right)+\left(1+\rho_1\rho_2\right)\right]^2-(\rho_1+\rho_2)^2}. \qquad (8.34')$$

Formula (8.34') is best given in the form:

$$\operatorname{ctg}^2\vartheta = \frac{\gamma\,(\rho_1+\rho_2)}{4-(\rho_2+\rho_1)^2} \qquad (8.34'')$$

From (8.27) it is also possible to obtain the relationship between the differential angular cross-section of the emitted particles *1* and *2* in the CM-system ($\sigma(\tilde{\vartheta})$), and in the L-system ($\sigma(\vartheta_1)$ and $\sigma(\vartheta_2)$). Proceeding from the conditions*

*Here, we shall also assume that the azimuthal distribution is isotropic, i.e. we shall not take into account the effects associated with polarization of the particles.

$$\sigma(\tilde{\vartheta})\,d\cos\tilde{\vartheta} = \sigma(\vartheta_n)\,d\cos\vartheta_n,$$

we have

$$\sigma(\tilde{\vartheta}) = \sigma(\vartheta_1)\,\gamma\,(1+\rho_1\cos\tilde{\vartheta})\,\frac{\sin^3\vartheta_1}{\sin^3\tilde{\vartheta}}, \qquad (8.35)$$

where in the right hand portion of (8.35) we can either express the angle $\tilde{\vartheta}$ in terms of ϑ_1, or alternatively, ϑ_1 by $\tilde{\vartheta}$. In the first case we again encounter expressions containing two signs ($\pm$) for $\rho_n > 1$ in front of the radical (for $\rho_n < 1$ it is only necessary to take the + sign, since for $\cos\tilde{\vartheta}$, also equal to $\frac{d\cos\tilde{\vartheta}}{d\cos\vartheta}$, there are ambiguous functions of the angles ϑ_n (see (7.18)). It is obvious that for $\vartheta_n = \vartheta_{n\,\max}$ (i.e. $\cos\vartheta_n = \min$) the value of $\frac{d\cos\tilde{\vartheta}}{d\cos\vartheta_n} = \infty$; and for two-branches of the function $\tilde{\vartheta}(\vartheta_n)$ the derivative $\frac{d\cos\tilde{\vartheta}}{d\cos\vartheta_n}$ has different signs, but just the + sign if the particle with the angle ϑ_n has the greatest (and the sign - in the case of the least) of the two possible values of momentum. Using for the expression $\tilde{\vartheta}(\vartheta_n)$ formulae (8.26), (8.30) and (8.33) we obtain for $\sigma(\vartheta_n)$ the relationship

$$\sigma(\vartheta_n) = \sigma(\tilde{\vartheta})\left(\frac{p_n}{\tilde{p}_K}\right)^2\frac{1}{\sqrt{1-\frac{\rho_n^2-V^2}{1-V^2}\sin^2\vartheta_n}}, \qquad (8.36)$$

which, in the special case of elastic scattering for particle *2*, originally at rest ($\rho_2 = 1$) becomes

$$\sigma(\vartheta_2) = \sigma(\tilde{\vartheta})\,\frac{4(1-V^2)\cos\vartheta_2}{(1-V^2\cos^2\vartheta_2)^2}. \qquad (8.36')$$

Using the substitution $\tilde{\vartheta}(\vartheta_1)$, we obtain from (8.35):

$$\sigma(\vartheta_1) = \sigma(\tilde{\vartheta})\,\frac{[\sin^2\tilde{\vartheta}+\gamma^2(\rho_1+\cos\tilde{\vartheta})^2]^{\frac{3}{2}}}{\gamma(1+\rho_1\cos\tilde{\vartheta})}, \qquad (8.37)$$

where ϑ_n is always a single-valued function of $\tilde{\vartheta}$. The direct relationship between differential cross-section of the two emitted reaction products - particles *1* and *2* - has the form:

$$\sigma(\vartheta_1) = \sigma(\vartheta_2)\,\frac{d\cos\vartheta_2}{d\cos\vartheta_1} =$$

$$= \left[-(1+\rho_1\rho_2) \pm \frac{(\rho_1+\rho_2)\operatorname{ctg}\vartheta_1}{\sqrt{\gamma^2(1-\rho_1^2)+\operatorname{ctg}^2\vartheta_1}}\right]\frac{\sin^3\vartheta_1}{\sin^3\vartheta_2}\,\frac{\sigma(\vartheta_2)}{1-\rho_1^2} =$$

$$=\frac{\sigma(\vartheta_2)\frac{\sin^3\vartheta_1}{\sin^3\vartheta_2}(1-\rho_2^2)}{\left[-(1+\rho_1\rho_2)\pm\frac{(\rho_1+\rho_2)\,\mathrm{ctg}\,\vartheta_2}{\sqrt{\gamma^2(1-\rho_1^2)+\mathrm{ctg}^2\vartheta_2}}\right]}. \qquad (8.38)$$

It is frequently necessary to associate the energy spectrum of the secondary particles of the reaction with their angular distribution in the CM-and L-systems. We shall make here a few preliminary observations, postponing a more detailed discussion until Section 12.

The relationship between the energy spectrum and the angular distribution is particularly simple in the CM-system. Actually. it is easy to obtain from relationship (8.26):

$$E_1=\frac{\widetilde{E}_1+V\widetilde{p}_\kappa\cos\widetilde{\vartheta}}{\sqrt{1-V^2}}, \qquad (8.26')$$

whence the Jacobian

$$\frac{dE_1}{d\cos\widetilde{\vartheta}}=\left|\frac{dE_2}{d\cos\widetilde{\vartheta}}\right|=\frac{V\widetilde{p}_\kappa}{\sqrt{1-V^2}}, \qquad (8.39)$$

i.e. for a given energy of the primary particle it has a constant value, independent of E_1 or E_2. Therefore the differential energy cross-sections (i.e. spectra) of particles *1* and *2* have the form:

$$\sigma(E_1)=\sigma(E_2=E_T-E_1)=2\pi\sigma(\widetilde{\vartheta})\left|\frac{d\cos\widetilde{\vartheta}}{dE_n}\right|= \\ =2\pi\sigma(\widetilde{\vartheta})\frac{\sqrt{1-V^2}}{V\widetilde{p}_\kappa}. \qquad (8.40)$$

It is obvious that the rectangular energy distribution of particles *1* and *2* from a minimum energy

$$(E_n)_{\min}=\gamma(\widetilde{E}_n-V\widetilde{p}_\kappa). \qquad (8.41)$$

up to the maximum energy

$$(E_n)_{\max}=\gamma(\widetilde{E}_n+V\widetilde{p}_\kappa). \qquad (8.41')$$

corresponds to the isotropy of the angular distribution in the CM-system.

It is necessary to emphasise that all these formulae for angular and energy distribution of the secondary particles *1* and *2* apply only if the energy of the bombarding particles I is some constant value. If, however, the bombarding particles have an energy spectrum then the general form of the angular and energy distribution of the reaction products is also a function of the total and differential cross-sections and the energies of the primary particles. In such circumstances, interpretation of the spectra and angular distributions experimentally observed is considerably complicated and frequently proves to be ambiguous.

Ultimately, we shall consider particular cases, but for the present we shall consider one problem, essential for calculations by the method of detailed balance.

We shall show here, without establishing the basis of this method (which is discussed in Chapter VIII) and the derivation of the formulae cited below, that between the differential (in the CM-system) and total cross-sections of the nuclear reaction $\mathrm{I}+\mathrm{II}\rightarrow 1+2$ and its inverse process $1+2\rightarrow \mathrm{I}+\mathrm{II}$, there is a relationship:

$$\frac{\sigma_{\mathrm{I\,II}}}{\sigma_{12}}=\frac{(2J_1+1)(2J_2+1)}{(2J_{\mathrm{I}}+1)(2J_{\mathrm{II}}+1)}\frac{\widetilde{p}_1^2}{\widetilde{p}_{\mathrm{I}}^2}, \qquad (8.42)$$

where J represents the spins of the interacting particles. The momenta of particles I and II in the direct reaction, and particles *1* and *2* in the inverse reaction should correspond of course, in the CM-system, to one and the same total energy of the interacting particles. From the condition of equality of the total energy in the CM-system, there immediately follows a relationship between the kinetic energy in the L-system of the bombarding particles in the direct and inverse reactions. Thus, if in the direct reaction in the L-system the particles I are moving and II are stationary, and in the inverse reaction in the L-system particles *1* are moving and *2* are stationary, then from (8.18) and (8.9) the relationship between the corresponding kinetic energies is:

$$W_1=\frac{m_{\mathrm{II}}}{m_2}W_{\mathrm{I}}+\frac{M_N+M_\kappa}{2m_2}Q=\frac{m_{\mathrm{II}}W_{\mathrm{I}}+MQ}{m_2}, \qquad (8.43)$$

where $M=\frac{1}{2}(M_N+M_\kappa)=M_N-\frac{Q}{2}$ (8.44)

It is clear that in all cases, except elastic scattering, either the direct reaction or the inverse reaction is endothermic, with a certain Q value. Therefore, relationship (8.43) can always be written in the form

$$W_1 = \frac{m_{II}}{m_2}(W_I - W_{I\,\mathrm{Thresh.}}) \qquad (8.45)$$

(if an endothermic direct reaction) or

$$W_I = \frac{m_2}{m_{II}}(W_1 - W_{1\,\mathrm{Thresh.}}) \qquad (8.45')$$

(for an endothermic inverse reaction). Further, by taking into account (8.5), (8.16) and (8.24) we obtain,

$$\frac{\sigma_{I\,II}(W_I)}{\sigma_{12}(W_1)} = \frac{(2J_1+1)(2J_2+1)}{(2J_I+1)(2J_{II}+1)} \frac{A_1^2 - 4m_1^2\left(M_n^2 + 2m_I\, W_I\right)}{4m_{II}^2 W_I (2m_I + W_I)}. \qquad (8.46)$$

We shall now give a few examples of the application of the formulae introduced above:

1) Photoproduction of π-mesons at nucleons (e.g. $\gamma + p \to \pi^+ + n$). The energy of the photon (ε_γ), the mass of the π-meson (π), its kinetic energy ω_π, and the total energy of the system $\tilde{\varepsilon}_T$ are expressed as units of the mass of the nucleon,

$$V = \frac{\varepsilon_\gamma}{1+\varepsilon_\gamma}, \qquad \gamma = \frac{1+\varepsilon_\gamma}{\sqrt{1+2\varepsilon_\gamma}}, \qquad \tilde{\varepsilon}_T = \sqrt{1+2\varepsilon_\gamma},$$

$$\rho_{1\,(\pi)} = \frac{\pi^2 + 2\varepsilon_\gamma}{\sqrt{(\pi^2 - 2\varepsilon_\gamma)^2 - 4\pi^2}} \frac{\varepsilon_\gamma}{1+\varepsilon_\gamma},$$

$$\rho_{2\,(n)} = \frac{2 + 2\varepsilon_\gamma + \pi^2}{\sqrt{(2 + 2\varepsilon_\gamma + \pi^2)^2 - 4\pi^2(1+2\varepsilon_\gamma)}} \frac{\varepsilon_\gamma}{1+\varepsilon_\gamma}.$$

The relationship between the kinetic energies for the direct and the inverse reaction of radiative capture of π-mesons by nucleons is

$$\omega_\pi = \varepsilon_\gamma - \pi\left(1 + \frac{\pi}{2}\right).$$

2) Photodisintegration of the deuteron: $\gamma + d \to p + n$ (we shall neglect here the discrepancy between the masses of two protons, two neutrons and the deuteron, since in the relativistic case the energies involved are much greater than

these discrepancies)

$$V = \frac{\varepsilon_\gamma}{2+\varepsilon_\gamma}, \qquad \gamma = \frac{2+\varepsilon_\gamma}{2\sqrt{1+\varepsilon_\gamma}}, \qquad \tilde{\varepsilon}_T = 2\sqrt{1+\varepsilon_\gamma},$$

$$\rho_1 = \rho_2 = \frac{\sqrt{\varepsilon_\gamma(1+\varepsilon_\gamma)}}{2+\varepsilon_\gamma}.$$

The energy relationship between the direct and inverse reaction is:

$$\omega_{p\,(n)} = 2\varepsilon_\gamma - \left(2 + \frac{q}{2}\right)|q|, \text{ where } q = \frac{Q}{m_{p\,(n)}}.$$

3) The production of a π^+-meson as a result of the collision of two protons:

$$p + p \to \pi^+ + d.$$

Here $m_I = m_{II} = m,\ m_1 = \pi m,\ m_2 = 2m,\ \omega_i = \frac{W_i}{m}$,

$$V = \sqrt{\frac{\omega_p}{2+\omega_p}}, \qquad \gamma = \sqrt{1 + \frac{\omega_p}{2}},$$

$$\rho_1 = \sqrt{\frac{\omega_p}{2+\omega_p}} \frac{2\omega_p + \pi^2}{\sqrt{(2\omega_p - \pi^2)^2 - 16\pi^2}},$$

$$\rho_2 = \sqrt{\frac{\omega_p}{2+\omega_p}} \frac{8 + 2\omega_p - \pi^2}{\sqrt{(8 + 2\omega_p - \pi^2)^2 - 32(2+\omega_p)}} \approx \frac{4+\omega_p}{\sqrt{\omega_p(2+\omega_p)}}$$

(because $\pi^2 \ll \omega_p$).

The energy relationship between the direct and inverse reaction is

$$2\omega_\pi = \omega_p - \pi\left(2 + \frac{\pi}{2}\right).$$

Comparison of the cross-sections of the direct and inverse reactions $p + p \rightleftarrows \pi^+ + d$ has established, as is well-known, that the spin of the π-meson is equal to zero. In this case, the relationship $\frac{\sigma_{pp}(\omega_p)}{\sigma_{\pi+d}(\omega_\pi)} = \frac{3}{2} \frac{(2\omega_p - \pi^2)^2 - 16\pi^2}{4\,\omega_p(2+\omega_p)}$ is valid, in which the factor $\frac{3}{2}$ should become $\frac{9}{2}$ or $\frac{15}{2}$ if the spin of the π-meson was equal to 1 or 2 respectively.

4) Elastic scattering of protons: $m_I = m_{II} = m_2 = m_2 = m$

$$\rho_1 = \rho_2 = 1, \quad V = \sqrt{\frac{\omega_p}{2+\omega_p}}, \quad \gamma = \sqrt{1+\frac{\omega_p}{2}}.$$

In this case, in place of (8.34) we obtain

$$\operatorname{ctg}\vartheta_1 \operatorname{ctg}\vartheta_2 = \gamma^2 = 1 + \frac{\omega_p}{2}.$$

Section 9. Basic Formulae for the Non-relativistic Case

Since nuclear reactions, in which the bombarding particles as well as the reaction products are moving with velocities much less than the velocity of light, are extremely widespread in laboratory practice, we shall repeat the derivations of the basic formulae of the preceding paragraph for the non-relativistic case.

Pointing out that the Q-values of reactions are usually very small compared with the masses of the interacting particles, we shall now assume $m_I + m_{II} = m_1 + m_2 = M$ and we shall consider not the total, but the kinetic energies of the particles. Then the laws of conservation of energy and momentum (for the stationary particle II) can be written in the form

$$W_I = W_1 + W_2 - Q. \tag{9.1}$$

$$\sqrt{2m_I W_I} = \sqrt{2m_1 W_1}\cos\vartheta_1 + \sqrt{2m_2 W_2}\cos\vartheta_2, \tag{9.2}$$

$$0 = \sqrt{2m_1 W_1}\sin\vartheta_1 - \sqrt{2m_2 W_2}\sin\vartheta_2, \tag{9.3}$$

whence

$$\sqrt{W_1} = \frac{\sqrt{m_I m_1 W_I}\cos\vartheta_1 \pm \sqrt{m_I m_1 W_I \cos^2\vartheta_1 + M[m_2 Q + (m_2 - m_I) W_I]}}{M}. \tag{9.4}$$

A similar expression - with the substitution of index 1 by 2 - is obtained for $\sqrt{W_2}$ (henceforth, in a number of cases we shall confine ourselves only to formulae for particle 1).

Frequently it proves more convenient to apply expression (9.4) in another form, viz:

$$Q = \frac{M}{m_2} W_1 - \frac{m_2 - m_I}{m_2} W_I - \frac{2\sqrt{m_I m_1 W_I W_1}}{m_2}\cos\vartheta_1. \tag{9.5}$$

From (9.4) it is easy to obtain the condition for the limiting angle of emission of the particles:

$$\sin^2\vartheta_1 \leqslant \frac{m_2}{m_1}\left\{\frac{M}{m_I}\left[1 - \frac{|Q|}{W_I}\right] - 1\right\} = \frac{m_2}{m_1}\left\{\frac{m_{II}}{m_I} - \frac{|Q|}{W_I}\left(1 + \frac{m_{II}}{m_I}\right)\right\}, \tag{9.6}$$

which, in the case of elastic scattering ($Q = 0$, $m_I = m_1$ and $m_{II} = m_2$ gives

$$\sin\vartheta_1 \leqslant \frac{m_{II}}{m_I}. \tag{9.7}$$

Assuming in (9.6) $\vartheta_{1\,\mathrm{Thresh.}} = 0$, we find an expression for the threshold of the endothermic nuclear reaction:

$$W_{I\,\mathrm{Thresh.}} = \frac{M}{m_{II}}|Q|. \tag{9.8}$$

In order to obtain the function $\vartheta_2 = f(\vartheta_1)$ it is also easier in this case to establish first the relationship between the angles of emission of the particles in the CM- and L-systems. This relationship is found from the simple law of addition of velocities (Figure 6):

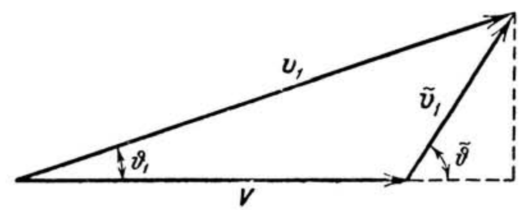


Fig. 6

Diagram of the relationship of the L- and CM-systems (non-relativistic case) - addition of velocities

$$\left.\begin{aligned} v_1\cos\vartheta_1 &= V + \tilde{v}_1\cos\tilde{\vartheta}, \\ v_1\sin\vartheta_1 &= \tilde{v}_1\sin\tilde{\vartheta}, \end{aligned}\right\} \tag{9.9}$$

where V - the relative velocity of the two co-ordinate systems is:

$$V = \frac{\mu_N}{m_{II}} v_I, \tag{9.10}$$

where $\mu_N = \frac{m_I m_{II}}{m_I + m_{II}}$ is the reduced mass of the system in the initial state. It is obvious that the momenta of the incident particles $\widetilde{p}_I = m_I\widetilde{v}_I = m_I(v_I - V)$ and $\widetilde{p}_{II} = m_{II}\widetilde{v}_{II} = m_{II}V$ satisfy the equation

$$\widetilde{p}_I = \widetilde{p}_{II} = \widetilde{p}_N = \frac{m_{II}}{M}\, p_I, \tag{9.11}$$

and their kinetic energies satisfy

$$\widetilde{W}_I = \frac{m_{II}^2}{M^2} W_I \text{ and } \widetilde{W}_{II} = \frac{m_I m_{II}}{M^2} W_I. \tag{9.12}$$

Hence the total kinetic energy of the primary particles in the CM-system is equal to

$$\widetilde{W}_N = \frac{m_{II}}{M} W_I = \frac{\mu_N}{m_I} W_I. \tag{9.13}$$

From (9.13) the expression for the threshold (9.8) follows.

Proceeding from equations $\widetilde{W}_T = \widetilde{W}_K - Q$ and $\widetilde{p}_1 = \widetilde{p}_2 = \widetilde{p}_K$ we obtain

$$\widetilde{p}_K = \sqrt{2\frac{\mu_K}{M}(m_{II}W_I + MQ)} \tag{9.14}$$

and

$$\widetilde{W}_1 = \frac{m_2}{m_1}\widetilde{W}_2 = \frac{m_2}{M^2}(m_{II}W_I + MQ), \tag{9.15}$$

where $\mu_K = \frac{m_1 m_2}{m_1 + m_2}$ - the reduced mass of the system in the final state.

Returning now to the relationships (9.9), we transform them in the following manner:

$$\operatorname{ctg}\vartheta_1 = \frac{\rho_1 + \cos\widetilde{\vartheta}}{\sin\widetilde{\vartheta}} \text{ and } \operatorname{ctg}\vartheta_2 = \frac{\rho_2 - \cos\widetilde{\vartheta}}{\sin\widetilde{\vartheta}} \tag{9.16}$$

or

$$\sin(\widetilde{\vartheta} - \vartheta_1) = \rho_1 \sin\vartheta_1 \tag{9.17}$$

and

$$\sin(\widetilde{\vartheta} + \vartheta_2) = \rho_2 \sin\vartheta_2, \tag{9.17'}$$

where

$$\rho_1 = \sqrt{\frac{m_1 m_I W_I}{m_2(m_{II}W_I + MQ)}}, \tag{9.18}$$

$$\rho_2 = \sqrt{\frac{m_2 m_I W_I}{m_1(m_{II}W_I + MQ)}}. \tag{9.18'}$$

The inverse function $\widetilde{\vartheta}$ from ϑ_1 and ϑ_2 has, in this case, the form

$$\sin\widetilde{\vartheta} = \rho_1 \sin\vartheta_1\cos\vartheta_1 \pm \sin\vartheta_1\sqrt{1 - \rho_1^2\sin^2\vartheta_1}, \tag{9.19}$$

$$\cos\widetilde{\vartheta} = -\rho_1\sin^2\vartheta_1 \pm \cos\vartheta_1\sqrt{1 - \rho_1^2\sin^2\vartheta_1}. \tag{9.19'}$$

The relationship between the angles ϑ_1 and ϑ_2 can be found by using relationship (8.34) for $\gamma = 1$.

We note in particular the special cases $\rho_2 = 1$ (elastic scattering: $m_I = m_1$, $m_{II} = m_2$) and $\rho_1 = \rho_2 = 1$ (elastic scattering of equal masses: $m_I = m_1 = m_{II} = m_2 = 1$). For $\rho_2 = 1$

$$\operatorname{ctg}\vartheta_1 = \frac{m_1 + m_2}{2m_2}\operatorname{tg}\vartheta_2 + \frac{m_1 - m_2}{2m_2}\operatorname{ctg}\vartheta_2, \tag{9.20}$$

and for $\rho_1 = \rho_2 = 1$, $\operatorname{ctg}\vartheta_1 = \operatorname{tg}\vartheta_2$ i.e. $\vartheta_1 + \vartheta_2 = \frac{\pi}{2}$. From (9.16) and the relationships (8.36) and (8.37) (for $\gamma = 1$), the relationship between the differential angular cross-sections in the L- and CM-systems is obtained.

The relationship between the energy spectrum of the secondary particles in the L-system and their angular distribution in the CM-system has, in the non-relativistic case, the form*:

$$\sigma(W_1) = \sigma(W_2) = 2\pi\sigma(\widetilde{\vartheta})\left|\frac{d\cos\widetilde{\vartheta}}{dW_{1,2}}\right| = 2\pi\sigma(\widetilde{\vartheta})\frac{M^2}{2\sqrt{m_I W_I m_1 m_2(m_{II}W_I + MQ)}}. \tag{9.21}$$

Transposing (9.5) we obtain

$$\cos\vartheta_1 = \frac{MW_1 - m_2 Q - (m_2 - m_I)W_I}{2\sqrt{m_I m_1 W_1 W_I}} \tag{9.22}$$

*This question is discussed in more detail in Section 12.

and

$$\frac{d\cos\vartheta_1}{dW_1}=\frac{MW_1+m_2Q+(m_2-m_1)W_I}{4W_1\sqrt{m_1m_1W_IW_1}} \qquad (9.23)$$

and a similar expression - with substitution of the index *1* by *2* - for the Jacobian $\frac{d\cos\vartheta_2}{dW_2}$. As a result, the relationship between the energy spectra and angular distribution in the L-system can be written in the form:

$$\sigma(W_n)=2\pi\cdot\frac{d\cos\vartheta_n}{d2W_n}\quad\sigma(\vartheta_n) \qquad (9.24)$$

with the consequent substitution of $\cos\vartheta_n$ in accordance with (9.22) or W_n in accordance with (9.4).

Section 10. Graphical Representation of Kinematic Relationships

The formulae deduced in the preceding paragraphs allow accurate calculations of the kinematic relationships necessary for the analysis of experiments to be made. Frequently, however, there is no need for great accuracy, and consequently a graphical representation proves to be useful; this allows rapid calculations dependent only on two initial parameters, all the basic kinematic relationships being obtained from the graph by means of a ruler and a protractor. The velocity of motion of the centre of mass in the L-system V and the momenta of the reaction products in the CM-system $\tilde{p}_1=\tilde{p}_2=\tilde{p}_\kappa$ (naturally for known masses m_I, m_{II}, m_1 and m_2), serve as the two initial parameters.

A detailed description of the graphical representation of kinematic relationships is given in a paper by Bleton [4]. This paragraph is devoted to an account of it.

a) Momentum Ellipse

As initial relationships we can re-write (8.26) in the form

$$p_{1x}=\gamma(\tilde{p}_x+V\tilde{E}_1),\quad p_{1y}=\tilde{p}_y, \qquad (10.1)$$

assuming that the vector of the total momentum of the system $p_T=p_1+p_2$ is directed along the x-axis.

From (10.1) it follows that

$$\tilde{p}_\kappa^2=p_{1y}^2+\left(\frac{p_{1x}}{\gamma}-V\tilde{E}_1\right)^2 \qquad (10.2)$$

or

$$\frac{p_{1y}^2+\frac{1}{\gamma^2}(p_{1x}-\gamma V\tilde{E}_1)^2}{\tilde{p}_\kappa^2}=1 \qquad (10.3)$$

It is easy to show that (10.3) represents the equation for an ellipse

$$\frac{p_{1y}^2}{b^2}+\frac{(p_{1x}-\alpha_1)^2}{a^2}=1 \qquad (10.4)$$

with the minor semi-axis

$$b=\tilde{p}_\kappa \qquad (10.5)$$

and with a focus of

$$f=\sqrt{a^2-b^2}=\sqrt{b^2(\gamma^2-1)}=\gamma V\tilde{p}_\kappa. \qquad (10.6)$$

The centre of the ellipse is displaced by a distance

$$\alpha_1=\gamma V\tilde{E}_1=\frac{f}{b}\tilde{E}_1 \qquad (10.7)$$

from the origin of the vector p_T of the total momentum.

Examination of the relationships for particle *2*, analagous to (10.1) leads to the conclusion that the extremity of the vector p_T is found on the opposite side of the centre of the ellipse at a distance

$$\alpha_2=\gamma V\tilde{E}_2=\frac{f}{b}\tilde{E}_2. \qquad (10.8)$$

From enumeration of all the characteristics of the ellipse, it is clear that for its construction it is necessary to know only the quantities $\tilde{p}_\kappa$ and V, defining the energy of the bombarding particles, and the equation for the nuclear reaction. The construction of one of the values of α_n for a known $\tilde{p}_\kappa$ and V is illustrated in Figure 7. Along the principal axis of the ellipse, to the left of its centre the value $\tilde{p}_\kappa=b$ is marked off, and at the point B_n a perpendicular is erected B_nK_n , with a length m_n . Then the centre of the ellipse is joined to the point K_n and the straight line OK_n produced beyond its intersection with

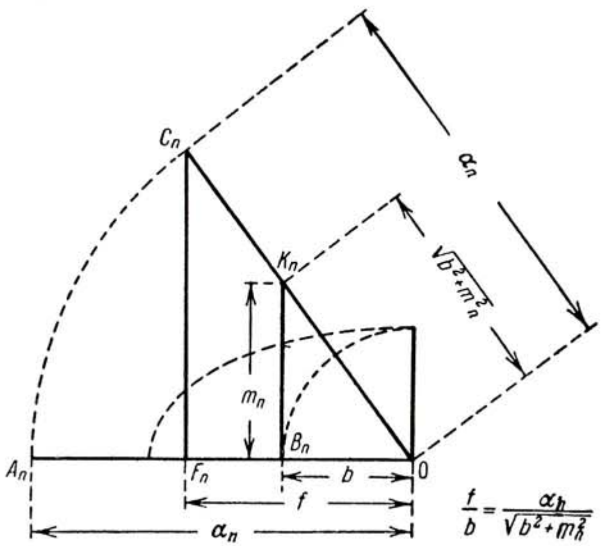


Fig. 7

Constructional diagram of the momentum ellipse

B_nK_n.. On the principal axis the quantity $f=\gamma Vb$, is also marked off and at the point F_n a perpendicular is erected which intersects at the point C_n with the straight line OK_n produced. It is obvious that $\frac{OC_n}{OF_n}=\frac{OK_n}{OB_n}$, i.e. $\frac{OC_n}{f}=\frac{\sqrt{b^2+m_n^2}}{b}$, and consequently

$$OC_n=\frac{f}{b}\sqrt{b^2+m_n^2}=\gamma V\widetilde{E}_n=\alpha_n.$$

Next, the segment OC_n of the circle is transferred onto the principal axis (OA_n). The ellipse itself is drawn using the given values of the semi-axes b and $a=\gamma b$ by any one of the standard methods of construction.

We note that all the parameters, necessary for construction of a momentum ellipse, are determined in the usual way, independent of whether the total momentum vector of the system p_T is formed as a result of the motion of both of the initial particles $(p_T=p_I+p_{II}$ or only of one of them$(p_T=p_I)$.

However, in all examples illustrating graphical construction, we shall consider the case when one of the primary particles (particle II, whereupon $m_I \leqslant m_{II}$) was stationary. For each of the particles it remains to discriminate between two cases: $\alpha_n > a$, i.e. the point A_n lies outside the ellipse, and $\alpha_n < a$, i.e. the point A_n lies inside the ellipse. Since $\frac{\alpha_n}{a}=\frac{V}{\widetilde{\beta}_n}=\rho_n$, then the case $\rho_n > 1$ or $\rho_n < 1$ corresponds to V the velocity of the CM-system relative to the L-system being greater than the velocity of the specified particles in the CM-system, or alternatively, to the velocity of the particle in the CM-system being greater than V. We note that at all times $\rho_n \geqslant V$, and that only for the reaction products with zero mass is $\rho_n = V$. For $\rho_n < 1$, particles corresponding to the index n, can be emitted through any angles, and these angles are single-valued and associated with the energies. If, $\rho_n > 1$, emission of the particles is possible only within the angular interval $0 \leqslant \vartheta_n \leqslant \vartheta_{n\,max}$, and for each angle, except $\vartheta_{n\,max}$, two values are possible for the energies of the particles.

From (8.32), (10.6) and (10.8) we obtain

$$\sin\vartheta_{n\,max}=\frac{b}{\sqrt{\alpha_n^2-f^2}}. \qquad (10.9)$$

Figure 8 shows an example of the graphical construction of kinematic characteristics (for the reaction $p+d\rightarrow\pi^+ + T$, having a Q = -135.5 MeV, for W_p = 660 Mev) by means of the momentum ellipse. It is obvious that the possible values and directions of the vectors p_1 and p_2 are determined by the position of the intercept drawn from the points A_1 and A_2 direct to the ellipse.

b) Relationships Between the L- and CM-systems

By means of the momentum ellipse, the inter-relationship between angles and momenta of particles *1* and *2* in the L- and CM-systems can be constructed graphically very simply. In point of fact, relationship (8.26) can obviously be written in the form

$$p_1\cos\vartheta_1 = a\cos\widetilde{\vartheta}+\alpha_1 \quad \text{and} \quad p_1\sin\vartheta_1 = b\sin\widetilde{\vartheta}. \qquad (10.10)$$

Having described on the graph two cricles with radii a and b and the momentum ellipse with these semi-axes, as indicated in Figure 9, the relationships between $\widetilde{p}$ and p_1, ϑ_1 and $\widetilde{\vartheta}$. can also be easily constructed. It is obvious, in parti-

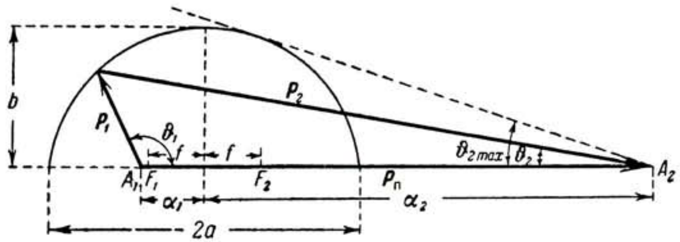


Fig. 8

Momentum ellipse for the reaction

$$p+d\rightarrow\pi^{+}+\mathrm{T}$$

$(1-\pi^{+},\ 2-\mathrm{T})$ for $W_p=660$ MeV.
Here $V=0.373,\ \gamma=1.078,\ b=365$ MeV, $a=393.5$ MeV.
$\alpha_1(\pi^{+})=157$ MeV $(\alpha_1<a)$, $\alpha_2(\mathrm{T})=1138$ MeV $(\alpha_2>a)$.
$f=146.8$ MeV. $\sin\vartheta_{2\,\max}=\frac{b}{\sqrt{a^2-f^2}}=0.324\ (\vartheta_{2\,\max}=18°50')$.

cular, that (as already mentioned above) in the case $\rho_n>1$, to the two possible values of p_n - for a given ϑ_n - there correspond also two possible angles $\tilde{\vartheta}_n$ in the CM-system. The energy of particles *1* and *2* in the L-system can also be obtained extremely simply from the momentum ellipse. According to (8.26'):

$$E_1=\frac{\alpha_1}{V}+f\cos\tilde{\vartheta}, \qquad (10.11)$$

i.e.

$$E_{1\,\min}=\frac{\alpha_1}{V}-f \qquad (10.12)$$

and

$$E_{1\,\max}=\frac{\alpha_1}{V}+f, \qquad (10.13)$$

$$E_{n\,\max}-E_{n\,\min}=2f. \qquad (10.14)$$

From (10.5), (10.7), (10.10) and (10.11), E_n can be represented in the form

$$E_n=\frac{b^2\alpha_n}{af}+Vp_n\cos\vartheta_n. \qquad (10.15)$$

The first term in (10.15) is constant and, consequently, the total energy of the particle emitted within the angle ϑ_n, is

determined by the product of the eccentricity of the ellipse $\varepsilon=f/a$ and the component of the momentum p_n along the axis p_T.

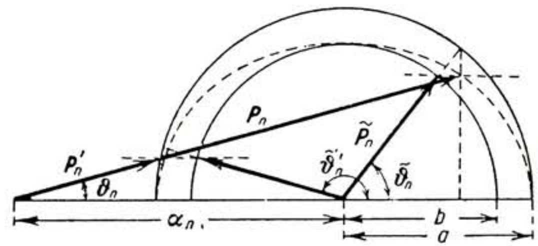


Fig. 9

Diagram of the relationship between angles and momenta of particles in the L- and CM-systems (relativistic case), by means of the momentum ellipse

Finally, it is necessary to note the particularly simple expression for the relationship between the energy spectrum of the secondary particles in the L-system and their angular distribution in the CM-system via the parameters of the momentum ellipse. In fact, (8.40) can be written in the following form

$$\sigma(E_n)=\frac{2\pi}{f}\sigma(\tilde{\vartheta}), \qquad (10.16)$$

where $E_n(\tilde{\vartheta})$ is defined by formula (10.11). As a result of the isotropy of scattering in the CM-system

$$\sigma(E_n)=\frac{\sigma_{\mathrm{Total}}}{2f}. \qquad (10.17)$$

c) Elastic Scattering

Proceeding to a consideration of the momentum ellipse for various special cases, we now note that particle II is initially stationary. Hence we can, by applying (8.24) and evaluating (8.9) and (8.44), write the following general expression for the minor semi-axis of the momentum ellipse:

$$b=\tilde{p}_k=\sqrt{\frac{(MQ+m_{\mathrm{II}}W_{\mathrm{I}})(MQ+m_{\mathrm{II}}W_{\mathrm{I}}+2m_1m_2)}{M_{\mathrm{N}}^2+2m_{\mathrm{II}}W_{\mathrm{I}}}}. \qquad (10.18)$$

For subsequent use we shall also write, from (8.15), the equations:

$$\frac{\sqrt{M_N^2+2m_{II}W_I}}{W_I+M_N}=\frac{1}{\gamma}, \tag{10.19}$$

$$\sqrt{\frac{W_I(W_I+2m_I)}{M_N^2+2m_{II}W_I}}=\gamma V. \tag{10.20}$$

$$\frac{(W_I+M_N)\sqrt{W_I(W_I+2m_I)}}{M_N^2+2m_{II}W_I}=\gamma^2 V. \tag{10.21}$$

It is obvious that the major semi-axis of the ellipse is

$$a=\frac{(W_I+M_N)\sqrt{(MQ+m_{II}W_I)(MQ+m_{II}W_I+2m_1m_2)}}{M_N^2+2m_{II}W_I}. \tag{10.22}$$

We shall consider elastic scattering, when $m_I=m_1$, $m_{II}=m_2$ and $m_I+m_{II}=M_N=m_1+m_2=M_k=M$, and in addition $\tilde{\beta}_{II}=\tilde{\beta}_2=V$, i.e. $\rho_2=1$. In this case

$$b=m_2\gamma V=\sqrt{\frac{m_2W_1(m_2W_1+2m_1m_2)}{M^2+2m_2W_1}}, \tag{10.23}$$

$$a=m_2\gamma^2V=\frac{(W_1+M)\sqrt{m_2W_1(m_2W_1+2m_1m_2)}}{M^2+2m_2W_1} \tag{10.24}$$

and according to (8.25), (8.18) and (10.7)

$$\alpha_1=a\frac{W_1+\frac{m_1}{m_2}M}{W_1+M}, \tag{10.25}$$

$$\alpha_2=a. \tag{10.26}$$

It is obvious that for $m_1 \geqslant m_2$, for elastic scattering $\rho_1 \geqslant 1$, and for $m_1<m_2$ $\rho_1<1$, i.e. a limiting angle of scattering is observed only for the scattering of a heavy particle by a light one.

Three possible cases of elastic scattering are illustrated respectively in Figs. 10, 11, 12: 1) $m_1=m_2$ (pp -scattering for W_p = 660 MeV), 2) $m_1>m_2$ (dp -scattering for $W_d=1$ BeV) and 3) $m_1<m_2$ (pd -scattering for $W_p=660$ MeV). By analysing the graphs represented by these diagrams and the corresponding formulae, a number of conclusions can be drawn:

1) The momentum of the initially stationary particle after scattering, in accordance with (8.33) and (10.23) is equal to

$$p_2=\frac{2m_2V\cos\vartheta_2}{1-V^2\cos^2\vartheta_2}, \tag{10.27}$$

i.e. its maximum momentum is equal to the principal axis of the ellipse

$$p_{2\max}=2a. \tag{10.28}$$

2) The kinetic energy of the initially stationary particle after scattering is, in accordance with (10.15)

$$W_2=\frac{b^2\alpha_2}{af}+Vp_2\cos\vartheta_2-m_2=Vp_2\cos\vartheta_2, \tag{10.29}$$

since for elastic scattering

$$\frac{b^2}{f}=m_2. \tag{10.30}$$

Thus, the kinetic energy is determined by the product of the horizontal component of the momentum p_2 and the eccentricity of the ellipse. It is obvious that

$$W_{2\max}=Vp_2(\vartheta_2=0)=2m_2\gamma^2V^2=2f. \tag{10.31}$$

3) From (10.27) and (10.29) it follows that

$$\cos\vartheta_2=\frac{1}{V}\sqrt{\frac{W_2}{W_2+2m_2}}, \tag{10.32}$$

i.e.

$$\mathrm{ctg}^2\vartheta_2=\frac{W_2}{2m_2V^2-\frac{W^2}{\gamma^2}},$$

and for a given value of the kinetic energy W_2, transferred to the stationary particle, at all times $\vartheta_2(W_2)<\theta_2$, thus

$$\mathrm{ctg}\,\theta_2=\sqrt{\frac{W_2}{2m_2}}. \tag{10.33}$$

Consequently, the maximum angle at which the target particle 2 can be emitted with a given kinetic energy, is independent of the mass of the energy of the bombarding particle 1. Therefore, in particular, the directions of the secondary

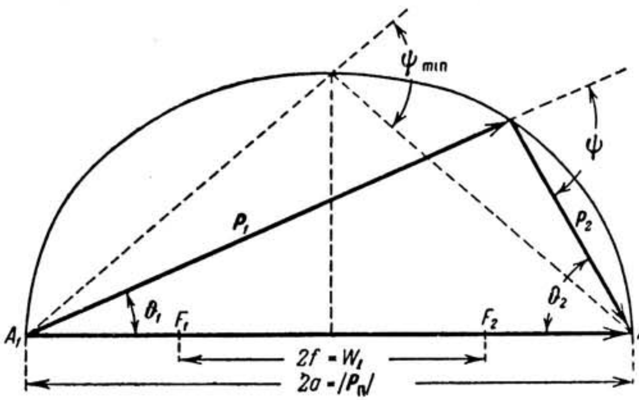


Fig. 10

Momentum ellipse for elastic pp-scattering for W_p = 660 MeV

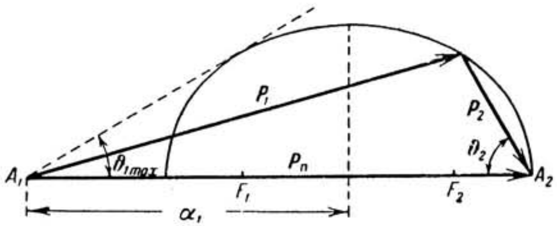


Fig. 11

Momentum ellipse for elastic dp-scattering for W_d = 1 BeV. Here, 1 is a deuteron, 2 is a proton $(m_1 > m_2)$.

particles in elastic collisions at relativistic energies $(W_2 \gg m_2)$, always make extremely small angles with the directions of the incident particles. For a given angle of emission of the secondary particle, ϑ_2 the inequality

$$\frac{W_2}{m_2} < 2\,\mathrm{ctg}^2\,\vartheta_2. \qquad (10.34)$$

always holds.

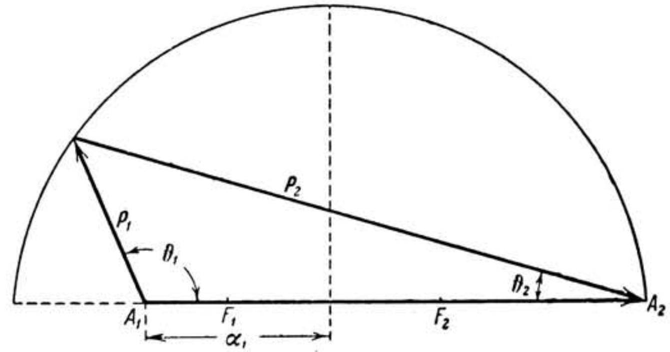


Fig. 12

Momentum ellipse for elastic pd-scattering for W_p = 660 MeV. Here, 1 is a proton, 2 is a deuteron $(m_1 < m_2)$

4) Knowing the direction of motion, the mass and the momentum of the secondary particle 2, the mass, m_I, of the incident particle can easily be established in terms of the known initial momentum p_I.

Actually, the total energy in the L-system

$$E_T = \frac{p_I}{V} = \frac{p_I p_2 \cos\vartheta_2}{W_2},$$

so that

$$m_I^2 = \left(\frac{p_I p_2 \cos\vartheta_2}{W_2} - m_2\right)^2 - p_I^2. \qquad (10.35)$$

5) As already mentioned above (Section 8), for elastic scattering of relativistic particles having equal rest masses, the kinetic energy of the incident particle can be determined from the angles ϑ_1 and ϑ_2, because

$$\mathrm{ctg}\,\vartheta_1\,\mathrm{ctg}\,\vartheta_2 = 1 + \frac{W_1}{2m}. \qquad (10.36)$$

The angle ψ between the directions of motion of the particles after scattering, for $m_1 > m_2$ can take any value from 0 to $\frac{\pi}{2}$

(whereupon $\sin\vartheta_{1\max}=\frac{m_2}{m_1}$), and for $m_1<m_2$ any value from $\frac{\pi}{2}$ to π . In the case of $m_1=m_2$, this angle is confined to the limits from $\psi_{\min}$ to $\frac{\pi}{2}$, where $\psi_{\min}=2\vartheta$ and $\vartheta_1=\vartheta_2=\vartheta$ so that

$$\cos\psi_{\min}=\frac{1}{1+\frac{4m}{W_{\rm I}}}. \qquad (10.37)$$

If one of the secondary particles of the reaction has a rest mass equal to zero, then $\widetilde{E}_n=\widetilde{p}_n=b$ and therefore, in accordance with (10.7), $\alpha_n=f$. Thus for zero rest mass the point A_n , from which the momentum of the corresponding particle is obtained, is located at one of the foci of the ellipse, and the absolute value of the momentum for different angles is given by the equation of the ellipse in polar co-ordinates.

For elastic scattering, the case $m_1=0$ corresponds to the Compton effect, i.e. to the scattering of photons with an initial momentum (and energy) $p_{\rm I}$ by particles with a mass m_2. For this it is obvious that

$$V=f=\frac{p_{\rm I}}{m_2+p_{\rm I}} \qquad (10.38)$$

and

$$p_1=\frac{m_2V}{1-V\cos\vartheta_2}. \qquad (10.39)$$

From these equations we obtain the well-known formula

$$p_1=\frac{p_{\rm I}}{1+\frac{p_{\rm I}}{m_2}(1-\cos\vartheta_2)}. \qquad (10.40)$$

As is apparent from (10.28) and (10.31), for Compton scattering

$$p_{\rm I}=a+f=\frac{1}{2}(p_{2\max}+W_{2\max}). \qquad (10.41)$$

d) Nuclear Reactions (Non-relativistic Case)

The graphical construction for non-relativistic nuclear interactions is considerably more simple than in the general case, because the momentum ellipse is reduced to a circle with radius $r=a=b$, determined by formula (9.14).

The values of ρ_n for this case are also determined above by expressions (9.18) and (9.18'), and the maximum angle of emission of the n -particles for $\rho_n\geqslant 1$ is given in accordance with (7.14) by the equation $\sin\vartheta_{n\max}=\frac{1}{\rho_n}$. For convenience we shall assume that $m_1\leqslant m_2$. It is obvious that

$$\sqrt{\frac{m_2W_{\rm I}}{m_{\rm II}W_{\rm I}+MQ}}<1$$

for exothermic reactions and greater than 1 for endothermic reactions, so that in the first case this expression increases, and in the second case decreases with increase of $W_{\rm I}$, converging within the limits to unity. Since in almost all reactions $m_{\rm I}\leqslant m_{\rm II}$, then in exothermic reactions in practice ρ_n is always less than 1($\rho_n<1$), i.e. for the direction of motion of the light particle there are in general no limitations. The direction of motion of a heavy particle can be limited only if $m_{\rm I}>m_1$, and if

$$W_{\rm I}\geqslant\frac{m_1}{m_{\rm I}-m_1}Q. \qquad (10.42)$$

For endothermic reactions the position is somewhat more complex. For these reactions, threshold $(W_{\rm I}=W_{\rm I\,Thresh.})$, the momentum circle is reduced to a point $(a=0$ and $\rho_1=\rho_2=\infty)$, and both particles move forward with momenta:

$$p_{1(2)\rm Thresh.}=m_{1(2)}\sqrt{\frac{2m_{\rm I}Q}{m_{\rm II}M}}=\frac{1}{M}\sqrt{2m_1m_2\frac{m_{1(2)}m_{\rm I}}{m_{2(1)}m_{\rm II}}MQ}. \qquad (10.43)$$

If

$$W_{\rm I}-W_{\rm I\,Thresh.}\geqslant\frac{m_1m_{\rm I}W_{\rm I\,Thresh.}}{M(m_2-m_1)}$$

then $\rho_1\leqslant 1$, so that the angles of emission of the light particle (I) are limited only for a small interval of the energy $W_{\rm I}$:

$$\frac{M}{m_{\rm II}}Q<W_{\rm I}<\frac{M}{m_{\rm II}}Q\frac{1-\frac{m_1}{M}}{1-\frac{m_{\rm I}}{m_2}}. \qquad (10.44)$$

As is apparent from (10.44) and (9.14), for $\rho_1=1$, the maximum possible momentum of the light particle is equal to

$$p_{1\,\max} = 2a = \frac{2}{M}\sqrt{2m_1m_2\left\{1+\frac{m_2}{M}\,\frac{M-m_I}{m_2-m_I}\right\}MQ}\,, \qquad (10.45)$$

which approximates to $2p_{1\,\mathrm{Thresh}}$.

The condition $p_2 \leqslant 1$ is possible only if $m_1 > m_I$ (for example in (nd), ($n\alpha$) etc. reactions) and if the following inequality holds,

$$W_I - W_{I\,\mathrm{Thresh.}} \gg \frac{m_2 m_I W_{I\,\mathrm{Thresh.}}}{M(m_1-m_I)}.$$

Thus, the angles of emission of the heavy particle (2) are limited for a somewhat larger interval of the energy W_I , viz:

$$\frac{M}{m_{II}}Q < W_I < \frac{M}{m_{II}}Q\,\frac{1-\dfrac{m_I}{M}}{1-\dfrac{m_I}{m_1}}. \qquad (10.46)$$

In Figures 13 and 14, the graphical representation is shown of the kinematics of the exothermic reaction $T^3(dn)He^4$ and its inverse endothermic reaction $He^4(nd)T^3$. Side by side with the method of construction of the momentum ellipse (here, a circle), already described above, a further one is shown, based upon the simple law of addition of velocities. In this method of construction, two circles are drawn with radii

$$\varkappa_{1(2)} = \sqrt{\frac{m_{2(1)}m_{II}}{M}}\sqrt{W_I - W_{I\,\mathrm{Thresh.}}} \qquad (10.47)$$

proportional to the velocity of the reaction products in the CM-system $\left(\varkappa_n = \sqrt{\frac{m_n}{2}}\,\tilde{v}_n\right)$. If the vectors $c_n = \sqrt{\frac{m_n}{2}}\,V$, be increased to $\varkappa_n$, proportional to $\vec{V}$ and equal in absolute value to $c_n = \frac{\sqrt{m_n m_I W_I}}{M}$, , then vectors are obtained equal in absolute value to the square root of the kinetic energy of the reaction products in the L-system.

<u>e) Photonuclear Interaction Processes</u>

In this particular case, $m_I = 0$ and $p_I = W_\gamma = E_\gamma = h\nu$. Therefor $m_{II} = M_N$, and the threshold equation for the reaction (8.13) takes the form:

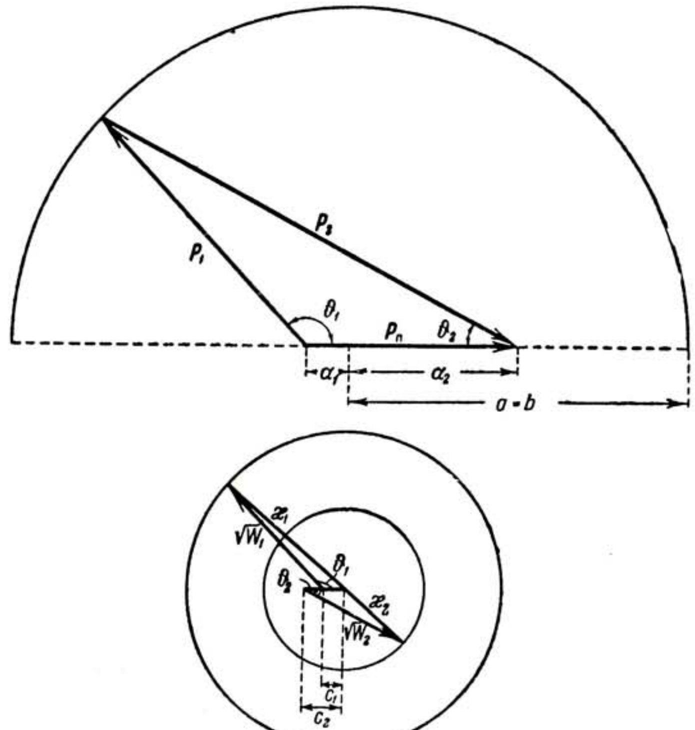


Fig. 13

Two methods of graphical construction of the kinematic characteristics of the reaction $T^3(dn)He^4$ for $W_d = 3$ MeV. Here 1 is a neutron, 2 is an α-particle. Q = 17.6 MeV

$$W_{I\,\mathrm{Thresh.}} = |Q|\left\{1+\frac{|Q|}{2M_N}\right\}. \qquad (10.48)$$

The minor axis of the ellipse

$$b = \tilde{p}_k = \sqrt{\frac{(W_\gamma - W_{\gamma\,\mathrm{Thresh}})\left(W_\gamma - W_{\gamma\,\mathrm{Thresh.}} + 2\frac{m_1m_2}{M}\right)}{1+2\frac{W_\gamma}{M}}}, \qquad (10.49)$$

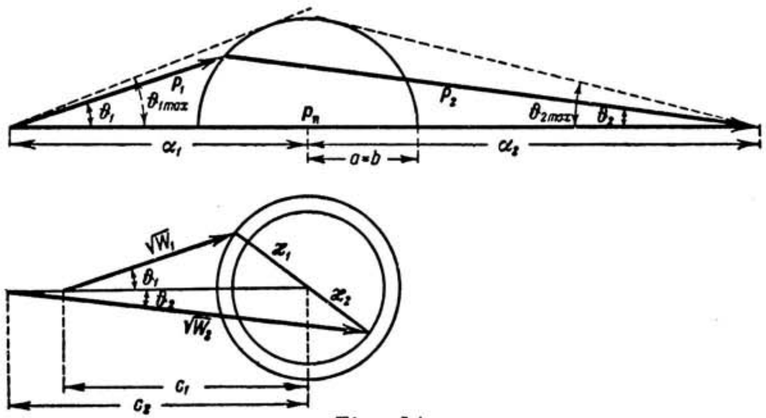


Fig. 14

Two methods of graphical construction of the kinematics of the reaction $He^4(nd)T^3$ for $W_I - W_{I\,Thresh} = 0.5$ MeV. Here I is a neutron, 1 is a deuteron, 2 is a triton

where

$$M = M_T - \frac{|Q|}{2}.$$

For $W_\gamma \ll M$ (10.49) the non relativistic relationship (9.14) is generally used, where $a = b$. We shall write (9.18) as a result of this in the form

$$p_{1(2)} = \frac{\alpha_{1(2)}}{a} = \sqrt{\frac{m_{1(2)} W_{I\,Thresh}}{2 m_{2(1)} M x}}(x+1). \qquad (10.51)$$

where the dimensionless parameter

$$x = \frac{W_I - W_{I\,Thresh}}{W_{I\,Thresh}}. \qquad (10.52)$$

It is easy to see that for $x = 0$, i.e. at threshold, $p_n = \infty$; subsequently p_n decreases rapidly with increase of x and approaches 1 for

$$x_{1(2)} \approx \frac{m_{1(2)} W_{I\,Thresh}}{2 m_{2(1)} M}.$$

The minimum values of p_1 and p_2 are attained for $x = 1$,

i.e. for $W_\gamma = 2W_{\gamma\,Thresh}$, when

$$p_{1(2)\,min} \approx \sqrt{\frac{2 m_{1(2)} W_{\gamma\,Thresh}}{m_{2(1)} M}} \ll 1, \qquad (10.53)$$

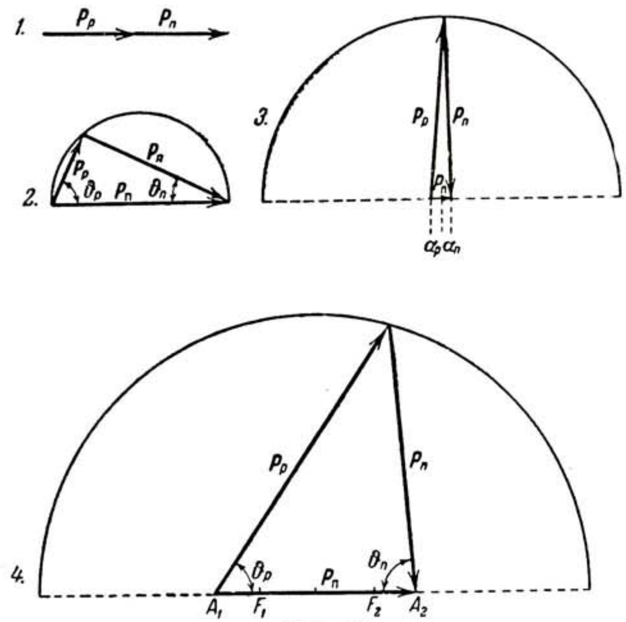


Fig. 15

Kinematics of the photodisintegration of the deuteron

$$\gamma + d \to p + n.$$

1) $W_\gamma = |p_n| = W_{Thresh}\, a = b = 0,\ p_p = p_n = \frac{p_n}{2}, \vartheta_p + \vartheta_n = 0;$

2) $W_\gamma = W_{Thresh}\left(1 + \frac{2W_{Thresh}}{m}\right),\ \alpha_p = \alpha_n = \frac{|p_T|}{2} \approx \frac{W_{Thresh}}{2};$
$\vartheta_p + \vartheta_n = 90°;$

3) $W_\gamma = 2W_{Thresh}\, \alpha_p = \alpha_n = \frac{|p_T|}{2} \approx W_{Thresh}\, \vartheta_p + \vartheta_n \approx 180°;$

4) $W_\gamma = 500$ MeV.

because $m_1 \sim m_2$ and $W_{\gamma\,\text{Thresh}} \ll M$. Examples of the graphical construction of photonuclear reactions for various energies of γ-quanta are shown in Figure 15 (photodisintegration of a deuteron) and in Figure 16 (photoproduction of π^+-mesons).

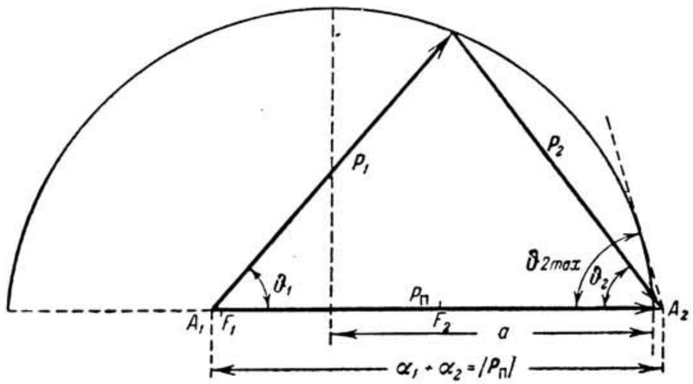


Fig. 16

Momentum ellipse for photoproduction of π^+-mesons at protons $\gamma + p \to \pi^+ + n\,(1-\pi^+,\ 2-n)$ for W_γ = 500 MeV

f) Decay of One Particle into Two

The kinematics of such disintegrations will be the subject of a special detailed discussion in a subsequent section. Here we shall dwell only upon the general methods of construction for the kinematic characteristics of this process.

In this special case $m_{II} = |p_{II}| = 0$, and the total energy in the CM-system is $\widetilde{E}_T = M_I = M_N = M$. It is obvious that the kinematics of the decay process are determined by the relationship between the kinetic energy of the primary particle in the L-system and its energy relationship in the CM-system, defined by the masses and energies of the secondary particles.

The minor semi-axis of the momentum ellipse in this case is equal to

$$b = \widetilde{p}_\kappa = \frac{1}{2M}\sqrt{\{M^2-(m_1+m_2)^2\}\{M^2-(m_1-m_2)^2\}}. \tag{10.54}$$

and the major semi-axis

$$a = b\frac{E_I}{M}. \tag{10.54'}$$

In addition, for the energies of the secondary particles in the CM-system we have

$$\widetilde{E}_{1(2)} = \frac{M^2 + m_{1(2)}^2 - m_{2(1)}^2}{2M} \tag{10.55}$$

and consequently

$$\alpha_{1(2)} = \frac{f}{b}\sqrt{b^2 + m_{1(2)}^2} = \frac{p_I}{M}\widetilde{E}_{1(2)} = \frac{p_I}{M}\left\{\frac{M^2 + m_{1(2)}^2 - m_{2(1)}^2}{2M}\right\}. \tag{10.56}$$

From the latter equation we obtain

$$\rho_{1(2)} = \frac{\alpha_{1(2)}}{a} = \frac{M^2 + m_{1(2)}^2 - m_{2(1)}^2}{\sqrt{\{M^2-(m_1+m_2)^2\}\{M^2-(m_1-m_2)^2\}}}\cdot\beta_I \tag{10.57}$$

The angle of emission of the n-particles is limited only if particle I has a sufficiently large kinetic energy.

$$p_I \geqslant p_{I\,\text{Thresh}} = \frac{1}{2m_n}\sqrt{\{M^2-(m_1+m_2)^2\}\{M^2-(m_1-m_2)^2\}} = b\frac{M}{m_n}, \tag{10.58}$$

consequently

$$\sin\vartheta_{n\,\max} = \frac{1}{2m_n p_I}\sqrt{\{M^2-(m_1+m_2)^2\}\{M^2-(m_1-m_2)^2\}}. \tag{10.59}$$

(10.58) is better given in the form:

$$E_I \geqslant E_{I\,\text{limit}}\ \frac{1}{2m_{1(2)}}\left[M^2 + m_{1(2)}^2 - m_{2(1)}^2\right]$$

and (10.59) in the form:

$$\sin\vartheta_{1(2)\max} = \frac{1}{2m_{1(2)}E_I}\left[M^2 + m_{1(2)}^2 - m_{2(1)}^2\right]$$

In Figures 17 and 18 two examples are given of the decay of one particle into two, in the first case $(\pi^0 \to 2\gamma)$ the rest mass of both products, and in the second case ($\pi^+ \to \mu^+ + \nu$) of one of them, is equal to zero. It is obvious that in the first of these examples

$$b = \frac{M}{2}, \quad a = \frac{E_I}{2}, \tag{10.60}$$

and

$$\alpha_1 = \alpha_2 = \frac{1}{2}\sqrt{E_I^2 - M^2} = \frac{1}{2}p_I = f. \tag{10.61}$$

In point of fact, it is obvious from (10.8) that for particles with zero rest mass, when $\widetilde{E}_n = b$ $\alpha_n = f$ at all times. Thus, for the decay of π^0-mesons the equation

$$\rho_1 = \rho_2 = \beta_I = \frac{p_I}{E_I} < 1,$$

holds good at all times, i.e. there is no limiting angle of emission of the photons in the L-system. In the second example

$$b = \frac{M^2 - \mu^2}{2M} \tag{10.62}$$

where μ is the mass of the μ-meson,

$$a = \frac{E_I}{2}\left(1 - \frac{\mu^2}{M^2}\right), \tag{10.63}$$

$$f = \frac{p_I}{2}\left(1 - \frac{\mu^2}{M^2}\right). \tag{10.64}$$

In this case

$$\alpha_\mu = f\,\frac{M^2 + \mu^2}{M^2 - \mu^2}, \tag{10.65}$$

and the kinetic energy of the μ-meson

$$\widetilde{W}_\mu = \frac{(M-\mu)^2}{2M} = 4.04 \text{ MeV};$$

for $E_I \geqslant \frac{M^2 + \mu^2}{2\mu}$, , i.e. $\widetilde{W}_I \geqslant 5.2$ MeV, the angle of emission of the μ-meson is limited. Finally, we observe that in accordance with (8.33) for a particle with zero rest mass

$$p_n(\vartheta_n) = \frac{b}{\gamma(1 - V\cos\vartheta_n)} = \frac{b^2}{a - f\cos\vartheta_n}. \tag{10.66}$$

Hence, using (8.36) we obtain for the angular distribution of n-particles in the L-system

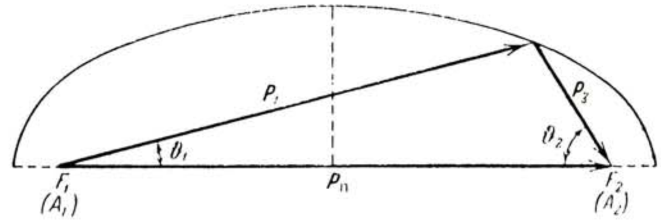


Fig. 17

Momentum ellipse for the decay $\pi^0 \to 2\gamma$ for $W_{\pi^0} = m_{\pi^0}$.

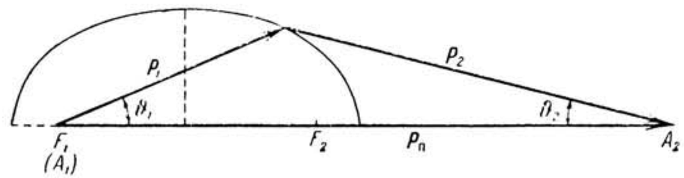


Fig. 18

Momentum ellipse for the decay

$\pi^+ \to \mu^+ + \nu\,(1 - \nu,\ 2 - \mu^+)$

for $W_{\pi^+} = \frac{3}{2} m_{\pi^+}$.

$$\sigma(\vartheta_n) = \frac{1}{4\pi}\,\frac{1}{\gamma^2(1 - V\cos\vartheta_n)^2} = \frac{1}{4\pi}\,\frac{b^2}{(a - f\cos\vartheta_n)^2}. \tag{10.67}$$

Thus, the probability of emission of an n-particle in the front hemisphere is obviously,

$$p\left(\frac{\pi}{2}\right) = \frac{1}{2}(1 + \beta). \tag{10.68}$$

Section 11. Decay into Two Particles

It has already been pointed out that the kinematic characteristics of decays into two particles are determined only by the values of the masses of the primary and of the secondary particles, and by two parameters, independent of the type of reaction (e.g. by the momentum of the primary particle and by the angle of emission of the secondary particle in the CM-system).

For an experimental investigation of decays into two particles, the following problems usually arise:

a) establishment of a criterion, to determine whether or not events observed in say photographic-emulsions or a Wilson cloud chamber are examples of the particular decay process;

b) determination of the mass of the primary particle, if the masses and momenta of both secondary particles are known;

c) determination of the mass of the secondary particle if the masses and momenta of the primary and the other secondary particles are known;

d) establishment of the relationship between the angular and energy distribution in the CM- and L-systems;

e) investigation of a criterion of accuracy of interpretation of the different events.

There are no practical methods for the analysis of decay processes, which give a simple indication that a specific individual case should be interpreted as a decay. For consideration of individual cases, the aim is usually to prove that it is impossible to identify them with other known processes (elastic or inelastic interaction of various types*).

In consequence of the circumstances noted, investigation of the existence of decay processes is generally based upon a statistical approach. In the first place, a small number of events which are interpreted as decay processes, are used to determine approximately its basic dynamic characteristics (type of decay, mass of the primary and secondary particles). Subsequently, for the analysis of a larger sample of events using a conjectural scheme, the approximate values of the characteristics, obtained earlier, are used. The conformity of the observed characteristics for an experimental sample of events with the conclusions resulting from the conjectural scheme of the decay process, verifies its accuracy. Naturally, for such a statistical approach, it is impossible to exclude the fact that a small number of cases, not fitting into the general picture, may have been included by mistake in the sample investiated.

*Later on (as is customary in works on high-energy physics) interaction in which a portion of the energy is expended on the formation of new particles will be called on inelastic interaction.

It stands to reason that the methods described are used not only to establish the occurence of a particular decay process, but also for control or the accuracy of the conjectural decay scheme.

In this section we shall employ the statistical approach for the analysis of decays into two particles. For this, naturally, it is assumed that the masses of the primary and secondary particles, and the type of decay process are fixed*. Furthermore, it is convenient to classify all modes of decay into two types: the first type, which we shall call the N-decay, represents conversion of a neutral particle into two charged particles; the second type (Z-decay) characterizes the decay of a charged particle into a charged and a neutral particle** (Figure 19).

Passing on to the investigation of the special features of these or other decay processes, we shall consider first of all the simplest case - decay of stationary particles. From formula (10.55), it follows that in this case the secondary particles possess constant energy, and consequently a range. Thus, the existence of a break with a constant track length from the point of break to the termination of the track (Z-decay), or the presence of characteristic bifurcations with constant values of both branches (N-decay) are clear indications that the decay of a stationary particle has taken place. It is precisely this circumstance that

*Certain methods of determination of the masses of the particles taking part in a decay process are presented below.

**We shall not dwell here on the decay of a neutral particle into two neutral particles. We note that in practice this boils down to the two determined previously, since no other methods of observation were available to us for observation of a neutral particle, except by study of their interaction or decay products.

led to the confirmation of the existence of the $\pi\mu$ decay.

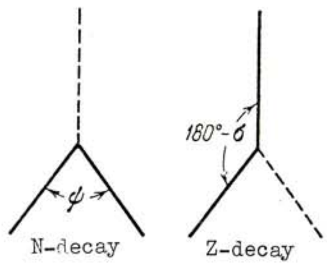


Fig. 19

Diagram of the decay of a neutral particle into two charged particles (N-decay), and for a charged particle into one neutral and one charged particle (Z-decay)

The study of decays of moving particles into two secondaries is more complex. The basis of one of the methods of establishing that such a decay has occured is the invariancy of the transverse components of the momenta p_T relative to a Lorentz transformation.

Actually, since the particles are moving in the CM-system isotropically, with one and the same momentum $\tilde{p}_K$, then the probability of emission of the particle within the interval $\cos\tilde{\vartheta}$, $\cos\tilde{\vartheta}+d\cos\tilde{\vartheta}$ is

$$\tilde{N}(\tilde{p}, \cos\tilde{\vartheta})\, d\tilde{p}\, d\cos\tilde{\vartheta} = \frac{1}{2}\delta(\tilde{p}-\tilde{p}_K)\sin\tilde{\vartheta}\, d\tilde{\vartheta}\, d\tilde{p} \qquad (11.1)$$

(δ is a delta function)

Since $\sin\tilde{\vartheta} = \frac{\tilde{p}_T}{\tilde{p}_K}$,

then

$$\tilde{N}(\tilde{p}_T)\, d\tilde{p}_T \sim \frac{\tilde{p}_T\, d\tilde{p}_T}{\tilde{p}_K\sqrt{\tilde{p}_K^2-\tilde{p}_T^2}}, \qquad (11.2)$$

where $\tilde{p}_K$ is determined by relationship (10.54).

Thus, if the events studied are examples of a decay into

two particles, then the distribution of the transverse components of the momenta complies with relationship (11.2). From this relationship it follows that the probability of the values of the momenta $\tilde{p}_T$ approaching $\tilde{p}_K$ is particula great.

A somewhat different idea of analysis is based on the introduction of the dimension α^*, the mean value of which depends only upon the invariant quantities. We shall determine the value of α^* in the following manner:

$$\alpha^* = \frac{p_{1\parallel}-p_{2\parallel}}{p_{1\parallel}+p_{2\parallel}}, \qquad (11.3)$$

where $p_{1\parallel}$ and $p_{2\parallel}$ are the longitudinal components of the momenta of both particles in the L-system. Since $p_{1\parallel}+p_{2\parallel}=p_1$, then

$$\alpha^* = \frac{p_1^2-p_2^2}{p_1^2}. \qquad (11.4)$$

Using (4.7), it follows that

$$\alpha^* = \frac{m_1^2-m_2^2}{M^2} + f(\tilde{p}_K, p_1)\cos\tilde{\vartheta}\ ^*). \qquad (11.5)$$

Summing over all directions in the CM-system, we finally obtain

$$\alpha^* = \frac{m_1^2-m_2^2}{M^2}. \qquad (11.6)$$

Measuring α^* by experiment, we can obtain in conjunction with (11.3), a relationship between the masses of the particles taking part in the reaction. This method was used especially for analysis of V- particle decays [6].

Another method of analysis of decays into two particles is based upon the existence of limiting values for the angles of emission of the secondary particles in the L-system. Generally speaking, the limiting values of the angles ϑ are determined by two parameters (for example, the magnitude of the momentum of the primary particle in the L-system and the momentum of the secondary particle in the CM-system). However, since for decays into two particles the momentum

*In the given case $M = m_1 (m_{II} = 0)$.

$\widetilde{p}_{\kappa}$ is entirely determined by the value of the masses (see (10.54)), in the given case the limiting relationship $\vartheta_{extr}(p_I)$ can be computed and, consequently, the limits can be determined for the angles of emission of secondary particles having a momentum p_I.

In one aspect of the analysis, which we shall apply to both types of decay (Z- and N-decay), we shall employ formula (7.14). Substituting (in 7.14) the values $\widetilde{\beta}_1 = \frac{\widetilde{p}_{\kappa}}{\sqrt{\widetilde{p}_{\kappa}^2 + m_1^2}}$ and $V = \frac{p_I}{\sqrt{p_I^2 + M^2}}$, we obtain

$$\vartheta_{1\,max} = \operatorname{arctg} \frac{\widetilde{p}_{\kappa} M}{\sqrt{p_I^2 m_1^2 - \widetilde{p}_{\kappa}^2 M^2}}, \tag{11.7}$$

if

$$p_I^2 m_1^2 > \widetilde{p}_{\kappa}^2 M^2,$$

and

$$\vartheta_{1\,max} = \frac{\pi}{2},$$

if

$$p_I^2 m_1^2 < \widetilde{p}_{\kappa}^2 M^2.$$

Consequently, for the decay of a particle with a mass M and momentum p_I into two secondary particles with mass m_1, it is not possible for the secondaries to deviate from the direction of the vector $\boldsymbol{p}_I$ by an angle greater than $\vartheta_{1\,max}$, as determined by formula (11.7).

We shall use the method involving the limiting values of the angles only in N-decay, it consists of the calculation of the limiting relationship $\vartheta_{extr}(p_I)$ (Figure 20). The data in this figure was obtained for the decay of a particle with a mass of 1000 m_e into two particles with masses of 280 m_e.

In point of fact, for fixed numerical values of the parameters, the angle between the directions of motion of the secondary particles is confined to specific intervals. Therefore, if in a large number of cases this angle lies within the calculated limits, then this is an important supplementary proof that the events under investigation are examples of a particular type of decay.

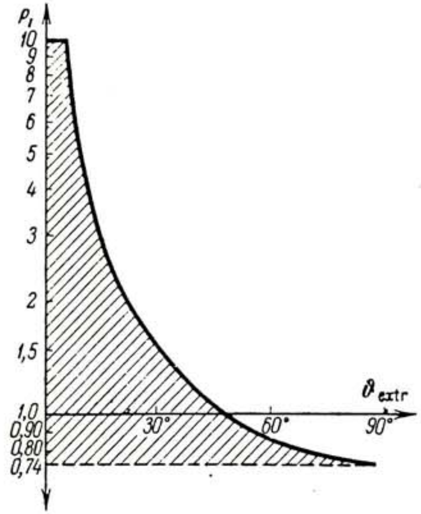


Fig. 20

Function $\vartheta_{extr}(p_I)$ for the decay of a particle with mass 1000 m_e into two particles with masses of 280 m_e

In conjunction with (4.1) it is possible to write down

$$\psi = \vartheta_1 + \vartheta_2 = \operatorname{arctg} \frac{1}{\gamma} \times \frac{\widetilde{p}_1 \sin \widetilde{\vartheta}_1}{\widetilde{p}_1 \cos \widetilde{\vartheta}_1 + V\widetilde{E}_1} + \operatorname{arctg} \frac{1}{\gamma} \frac{\widetilde{p}_2 \sin \widetilde{\vartheta}_2}{\widetilde{p}_2 \cos \widetilde{\vartheta}_2 + V\widetilde{E}_2}. \tag{11.8}$$

Bearing in mind that $\vartheta_2 = \pi - \vartheta_1$ and confining oneself, for simplicity, to analysis of the case when $m_1 = m_2 = m$, it is possible to write (11.8) into the following form:

$$\psi = \operatorname{arctg} \frac{1}{\gamma} \frac{\widetilde{p}_{\kappa} \sin \widetilde{\vartheta}_1}{\widetilde{p}_{\kappa} \cos \widetilde{\vartheta}_1 + V\widetilde{E}_{\kappa}} + \operatorname{arctg} \frac{1}{\gamma} \frac{\widetilde{p}_{\kappa} \sin \widetilde{\vartheta}_1}{V\widetilde{E}_{\kappa} - \widetilde{p}_{\kappa} \cos \widetilde{\vartheta}_1}, \tag{11.9}$$

where $\widetilde{p}_{\pi}=\sqrt{\frac{M^2-4m^2}{4}}$. The function $\psi(\cos\widetilde{\vartheta}_1)$ has limiting values for $\cos\widetilde{\vartheta}_1=0$ (maximum), and also (minimum) for

$$\cos\widetilde{\vartheta}_1=\frac{1}{p_I}\sqrt{\frac{p_I^2M^2-8p_I^2m^2+M^4-4m^2M^2}{M^2-4m^2}}. \tag{11.10}$$

Dependent upon the sign of the expression M^2-8m^2 , equation (11.10), determining the minimum value of ψ_{min} , has a solution within the range $(0<\cos\vartheta<1)$ for different values of p_I if

a) $$M^2-8m^2>0.$$

Equation (11.10) has a solution for

$$p_I^2>\frac{M^2(M^2-4m^2)}{4m^2}$$

if b) $$M^2-8m^2<0.$$

Equation (11.10) has a solution for

$$\frac{M\sqrt{M^2-4m^2}}{2m}<p_I<M\sqrt{\frac{M^2-4m^2}{8m^2-M^2}}.$$

In Figure 21 are shown curves of $\psi_{extr}(p_I)$ for the case of N-decay of particles with a mass 1000 m_e into two charged π -mesons (m_π has been taken equal to 280 m_e). The cross-hatched area corresponds to permitted values of the angles ψ .

The method described was used by Batler [7] for analysis of the decay scheme of Λ^0- and θ^0 - particles.

One should also consider another criterion of analysis, the method of complanarity. However this method, has the disadvantage that it applies not only to a decay process but also to elastic scattering, and since the values of the masses are not used, it cannot be recommended as proof of t the decay nature of a process. Conversely, if it is already established that a given event is a decay process, then the condition for complanarity can serve as a criterion for establishing the number of secondary particles.

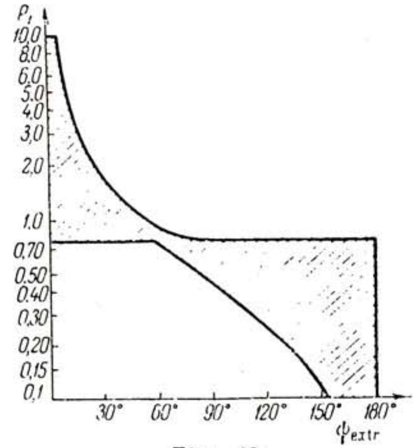


Fig. 21

Limiting relationship $\psi_{extr}(p_I)$ for the decay of particles with mass 1000 m_e into two particles with masses of 280 m_e

The criterion of complanarity is based upon the simple relationship

$$\boldsymbol{p}_I=\boldsymbol{p}_1+\boldsymbol{p}_2. \tag{11.11}$$

Multiplying this vector equation scalarly by $\boldsymbol{n}$ ($\boldsymbol{n}$ is a vector, normal to the plane formed by two of the three vectors, e.g. $\boldsymbol{p}_I$ and $\boldsymbol{p}_1$), we obtain $(\boldsymbol{n}\boldsymbol{p}_2)=0$, i.e. that the three vectors also lie in the same plane.

One should note that in certain practical cases, e.g. when the mass of one of the secondary particles approaches the mass of the primary particle, the methods indicated above can lead to a mistaken identification of events as decays into two particles, because a larger number of particles may have participated in the processes. These special cases are discussed in more detail in [8].

Let us derive further formulae which determine the masses of interacting particles.

a) Determination of the mass M of the primary particle by the kinematic characteristics. From the equations of conservation

$$\mathbf{p}_I = \mathbf{p}_1 + \mathbf{p}_2, \tag{11.12}$$

$$E_I = E_1 + E_2 \tag{11.13}$$

we obtain

$$M^2 = m_1^2 + m_2^2 + 2\{[(p_1^2+m_1^2)(p_2^2+m_2^2)]^{1/2} - p_1p_2\cos\psi\}. \tag{11.14}$$

If the decay of a stationary particle is considered, ($p_1 = p_2 = p_x$ and $\psi = \pi$), then formula (11.14) is considerably simplified:

$$M^2 = m_1^2 + m_2^2 + 2[(p_x^2+m_1^2)(p_x^2+m_2^2)]^{1/2}. \tag{11.15}$$

Equation (11.14) is also simplified if $p_1 \ll m_1$, or $p_2 \ll m_2$, whence the term $p_1p_2\cos\psi \ll [(p_1^2+m_1^2)(p_2^2+m_2^2)]^{1/2}$, and it can be neglected.

Sometimes the mass of the primary particle is required, the energy of one of the secondary particles and its angle of emission being known [9]. In conjunction with (4.7) one can write

$$M = \gamma\left\{(E_1 - p_1V\cos\vartheta_1) + \sqrt{(E_1 - p_1V\cos\vartheta_1)^2 - \frac{m_1^2 - m_2^2}{\gamma^2}}\right\}. \tag{11.16}$$

Actually, if $m_2 = 0$, then

$$M = \tilde{E}_1 + |\tilde{\mathbf{p}}_1|. \tag{11.17}$$

In this case, if $\cos\vartheta_1 = \pm 1$, then

$$M = \gamma\{E_1 \mp p_1V + |p_1 \mp E_1V|\}. \tag{11.18}$$

If $\cos\vartheta_1 = 1$, then dependent upon the sign of the expression $p_1 + E_1V$, formulae (11.17) and (11.18) can be written in the following manner:

$$M = \sqrt{\frac{1-V}{1+V}}[E_1 + p_1] \quad \left(\text{for } \frac{p_1}{E_1} > V\right), \tag{11.19}$$

$$M = \sqrt{\frac{1+V}{1-V}}[E_1 - p_1] \quad \left(\text{for } \frac{p_1}{E_1} < V\right). \tag{11.20}$$

If $\cos\vartheta_1 = -1$, then at all times

$$M = \sqrt{\frac{1+V}{1-V}}[E_1 + p_1]. \tag{11.21}$$

b) Determination of the mass of a secondary particle (Z-decay). From equation (11.12), (11.13) it follows that

$$m_2^2 = M^2 + m_1^2 - 2\{[(M^2+p_I^2)(m_1^2+p_1^2)]^{1/2} - p_Ip_1\cos\vartheta_1\}. \tag{11.22}$$

Similarly, as earlier, if $p_I \ll M$ or $p_1 \ll m_1$, then the term $p_Ip_1\cos\vartheta_1$ can be neglected. Actually, if stationary particles are decaying, then

$$m_2^2 = M^2 + m_1^2 - 2M\sqrt{m_1^2 + p_1^2}. \tag{11.23}$$

Section 12. Relationship between Angular and Energy Distribution of Secondary Particles in the CM-system and the L-system*.

In this paragraph we shall consider a number of problems concerning decay into two-particles, when the momenta $\tilde{p}_1$ and $\tilde{p}_2$ are equal so that $\tilde{p}_1 = \tilde{p}_2 = \tilde{p}_x$ and can be determined from equation (10.54). Consequently, a general expression can be obtained for the angular and energy distribution by applying the formulae obtained in the preceding chapter.

We shall begin with a consideration of the angular distribution of secondary particles for the simplest case, when the primary particles have a definite energy and are moving in the same direction.

In the CM-system, we shall assume the angular distribution of the secondary particles to be isotropic and that the particles possess a constant momentum. Consequently, the probability of emission of the particle at an angle, which must fall within the interval between $\cos\tilde{\vartheta}$ and $\cos\tilde{\vartheta} + d\cos\tilde{\vartheta}$,

*We shall consider the transformation of an isotropic decay for a constant momentum, since this simple case plays an important role for analysis of collisions (See Chapter V).

is determined by equation (11.1). The angular distribution $N(\vartheta)\,d\cos\vartheta$ therefore has the following form:

$$N(\vartheta)\,d\cos\vartheta=\frac{1}{2}\,d\cos\vartheta\int\limits_{p_{\min}}^{p_{\max}}\delta\left[\widetilde{p}(p)-\widetilde{p}_{\kappa}\right]J\,dp. \qquad (12.1)$$

where J is determined by formula (7.2). By using the formula,

$$\delta\left[\widetilde{p}(p)-\widetilde{p}_{\kappa}\right]=\sum_{m}\frac{\delta(p-p_m)}{\left|\frac{d\widetilde{p}}{dp}\right|_{p=p_m}} \qquad (12.2.)$$

(p_m is the radical of the equation $\widetilde{p}(p)=\widetilde{p}_{\kappa}$ within the interval for which the integration is carried out) and taking into account equations (4.7), (7.4) and (7.12), (12.1) can be written in the following form:

$$N(\vartheta)\,d\cos\vartheta=\frac{p_m^2\,d\cos\vartheta}{2\gamma\widetilde{p}_{\kappa}\left[p_m-E_mV\cos\vartheta\right]}, \qquad (12.3.)$$

if $\widetilde{\beta}>V$,

where E_m is the energy corresponding to the momentum p_m and

$$N(\vartheta)\,d\cos\vartheta=\frac{d\cos\vartheta}{2\gamma\widetilde{p}_{\kappa}}\left[\frac{p_{m_1}^2}{p_{m_1}-E_{m_1}V\cos\vartheta}+\frac{p_{m_2}^2}{E_{m_2}V\cos\vartheta-p_{m_2}}\right], \qquad (12.4)$$

if $\widetilde{\beta}<V$ and $p_{m_1}>p_{m_2}$.

Using the right-hand portion of relationship (7.12) as a solution to the equation $\widetilde{p}(p)-\widetilde{p}_{\kappa}=0$, we obtain

$$N(\vartheta)\,d\cos\vartheta=\frac{\left[\widetilde{E}_{\kappa}V\cos\vartheta+\sqrt{m^2\gamma^2V^2\cos^2\vartheta+\widetilde{E}_{\kappa}^2-m^2\gamma^2}\right]^2d\cos\vartheta}{2\widetilde{p}_{\kappa}\gamma^2(1-V^2\cos^2\vartheta)^2\sqrt{m^2\gamma^2V^2\cos^2\vartheta+\widetilde{E}_{\kappa}^2-m^2\gamma^2}}, \qquad (12.5)$$

if $\widetilde{\beta}>V$. varies within the interval 0 to π . By $\widetilde{E}_{\kappa}$, here and henceforth, is understood the energy of one or other secondary particle in the CM-system (in the final state). Since $\widetilde{E}_{\kappa}$ is related to the mass of the particle and its momentum $\widetilde{p}_{\kappa}$ (identical for both secondary particles) by the relationship $\widetilde{E}_{\kappa}=\sqrt{\widetilde{p}_{\kappa}^2+m^2}$, bearing in mind that

$\widetilde{E}_{\kappa}$ and m relate to the same particle, we shall drop the indices 1, 2 or n here and henceforth.

Actually, for elastic collisions, when $m_{\rm I}=m_{\rm II}$, $\widetilde{E}_{\kappa}=\gamma m$, we obtain

$$N(\vartheta)=\frac{2\cos\vartheta}{\gamma^2(1-V^2\cos^2\vartheta)^2}. \qquad (12.6)$$

If $\widetilde{\beta}<V$, then

$$N(\vartheta)\,d\cos\vartheta=\frac{\left[\widetilde{E}_{\kappa}^2V^2\cos^2\vartheta+m^2\gamma^2V^2\cos^2\vartheta+\widetilde{E}_{\kappa}^2-m^2\gamma^2\right]d\cos\vartheta}{\widetilde{p}_{\kappa}\gamma^2(1-V^2\cos^2\vartheta)^2\sqrt{m^2\gamma^2V^2\cos^2\vartheta+\widetilde{E}_{\kappa}^2-m^2\gamma^2}}. \qquad (12.7)$$

In the latter case ϑ varies within the interval 0 to $\vartheta_{\max}$ [$\vartheta_{\max}$ is determined by relationship (7.14)].

Formula (12.5) is simplified if $\widetilde{E}_{\kappa}\gg m\gamma$; whence

$$N(\vartheta)\,d\cos\vartheta=\frac{d\cos\vartheta}{2\gamma^2(1-V\cos\vartheta)^2}. \qquad (12.8)$$

For this case it is easy to determine the mean angle $\bar{\vartheta}$:

$$\bar{\vartheta}=-\frac{1}{2\gamma^2}\int\limits_0^{\pi}\frac{\vartheta\,d\cos\vartheta}{(1-V\cos\vartheta)^2}=\frac{\pi}{2}\,\frac{1}{\gamma^2V}\left[\gamma-\frac{1}{1+V}\right]. \qquad (12.8')$$

If $\gamma\gg1$, then $\bar{\vartheta}=\frac{\pi}{2\gamma}$.

We shall use the formulae presented in this section for analysis of decays into two particles. In addition to this, for determination of the parameter of scattering $\widetilde{p}_{\kappa}$ it will be necessary to use formula (10.54). The angular distribution of particles with mass m_1 can be written down in the form

$$N(\vartheta_1)\,d\cos\vartheta_1=$$

$$=\frac{\left[(M^2+m_1^2-m_2^2)V\cos\vartheta_1+\sqrt{4M^2m_1^2V^2\gamma^2\cos^2\vartheta_1+(M^2+m_1^2-m_2^2)^2-4M^2m_1^2\gamma^2}\right]^2}{2\gamma^2(1-V^2\cos^2\vartheta_1)^2\sqrt{M^4+m_1^4+m_2^4-2(M^2m_1^2+M^2m_2^2+m_1^2m_2^2)}}\times \qquad (12.9)$$

$$\times\frac{d\cos\vartheta_1}{\sqrt{4M^2m_1^2V^2\gamma^2\cos^2\vartheta_1+(M^2+m_1^2-m_2^2)^2-4M^2m_1^2\gamma^2}},$$

if $\widetilde{\beta}>V$, and similarly for $\widetilde{\beta}<V$. Formula (12.9) takes a particularly simple form in the case of decay into two relativistic particles. For example, for decay of a particle into two photons (such an example is the decay of a

π^0-meson), formula (12.6) can be used for the angular distribution.

For example, Figure 22 shows the angular distribution of the secondary particles formed as a result of the decay of a Λ-particle into two secondaries, according to the scheme $\Lambda \rightarrow p + \pi^-$.

We shall consider the evaluation of the energy distribution for the simplest case of a mono-energetic stream of primary particles.

The momentum distribution $N(p)\,dp$ is determined by the integral

$$N(p)\,dp = \frac{dp}{2}\int_0^{\vartheta_{max}} \delta\left[\widetilde{p}(\vartheta) - \widetilde{p}_{\kappa}\right] J \sin\vartheta\, d\vartheta,$$

where J is the Jacobian, calculated according to formula (7.2). Using formulae (4.7), (7.4) and (12.6) it is easy to obtain

$$N(p)\,dp = \frac{p\,dp}{2E\widetilde{p}_{\kappa}\gamma V}. \qquad (12.10)$$

p varies within the limits

$$\gamma\,|\widetilde{p}_{\kappa} - \widetilde{E}_{\kappa}V| \leqslant p \leqslant \gamma(\widetilde{p}_{\kappa} + \widetilde{E}_{\kappa}V). \qquad (12.11)$$

The latter formulae acquire greater significance if charged from momentum distribution to energy distribution. Using the relationship $p\,dp = E\,dE$, we obtain

$$N(E)\,dE = \frac{dE}{2\widetilde{p}_{\kappa}\gamma V}. \qquad (12.12)$$

Thus, as already mentioned (see Section 10), in discussing the case of decay into particles, the energy distribution is uniform within the interval

$$\gamma\,|\widetilde{E}_{\kappa} - \widetilde{p}_{\kappa}V| \leqslant E \leqslant \gamma(\widetilde{E}_{\kappa} + \widetilde{p}_{\kappa}V). \qquad (12.13)$$

Outside of this interval $N(E) = 0$. This relationship has an intrinsic significance in finding a relationship between the characteristics of the energy spectra of primary and secondary particles. It is extremely important because the value $\widetilde{E}_{\kappa}$ occurs in (12.13) and it is necessary to establish certain general inherent tendencies of the spectra as

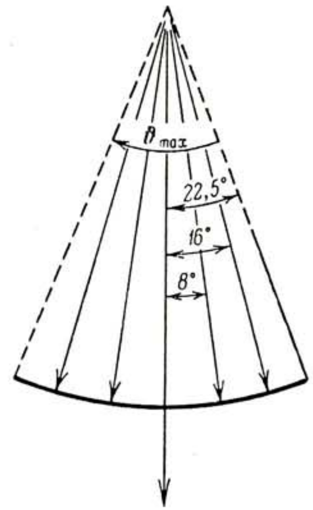


Fig. 22

Function $N(\vartheta)$ for decay of a Λ -particle according to the scheme

a function of the values of the masses of the primary and secondary particles.

We shall begin with an examination of the simplest case of decay into two particles with zero rest mass [10] (e.g. $\pi^0 \rightarrow 2\gamma$). These particles can have any energy within the interval from $\frac{M}{2}\sqrt{\frac{1-V}{1+V}}$ to $\frac{M}{2}\sqrt{\frac{1+V}{1-V}}$. All the remaining values of the energy are found to be forbidden.

From the expressions determining the limits of the energy, it follows that whatever may have been the energy possessed by the primary particle, the secondary particle can always possess an energy equal to $\frac{M}{2}$. Let us consider the spectrum of the primary particles extending from 0 to ∞ . For any specified energy of the primary particle, the secondary particles are distributed uniformly within certain energy limits and can possess an energy $\frac{M}{2}$. Therefore, for this value, independent of the form of the infinite spectrum of the primary particles, a maximum in the energy spectrum for the secondary particles will be observed. Let us consider two values of energy E_1 and E_2, having the property that $N(E_1)=N(E_2)$. Since the spectrum of the secondary particles has a maximum for any distribution of the primary particles, then at all times it has an infinite number of such pairs of energy values, disposed on either side of the maximum. It is clear that E_1 and E_2 should coincide with the limits of the allowed energy interval corresponding to some specific energy of the primary particle $\frac{M}{\sqrt{1-V^2}}$.

Therefore, it is possible to write

$$\left.\begin{aligned}\frac{M}{2}\sqrt{\frac{1-V}{1+V}}=E_1,\\ \frac{M}{2}\sqrt{\frac{1+V}{1-V}}=E_2\end{aligned}\right\} \quad (12.14)$$

and consequently

$$M=2\sqrt{E_1E_2}. \quad (12.15)$$

Let us consider the general case of decay into two particles of arbitrary mass. Here, it can be proved at once that in the case of an arbitrary spectrum for the primary particles, an inherent tendency similar to that described above, does not exist. Furthermore, if $V\to 0$, then the limiting value of the energy is $E_1\to E_2\to \widetilde{E}_\kappa$. Consequently a possible condition for a maximum in the spectrum of the secondary particles should be found in the vicinity of the value $E=\widetilde{E}_\kappa$. If $V\to 1$ then at all times a value of V_1 is found (and consequently also of γ), at which the left hand limit of the interval $\gamma(\widetilde{E}_\kappa-\widetilde{p}_\kappa V$ is greater than the value of $\widetilde{E}_\kappa$ *. Therefore, for certain $V>V_1$, the value of E_κ will lie outside the permissible interval, and therefore will not occur. As an illustration we shall consider the extreme case when $\widetilde{p}_\kappa$ is negligibly small compared with the magnitude of $\widetilde{E}_\kappa$ (such a case is found, for example, in the analysis of the spectrum of the protons arising from the decay of Λ^0-particles). Then at all times $E_1\approx E_2\approx\gamma\widetilde{E}_\kappa$,

*We recall that the case $\widetilde{E}_\kappa\neq\widetilde{p}_\kappa$ is considered.

and therefore the spectrum of the secondary particles is a repetition of the spectrum of the primary particles.

Another situation arises, however, in the case where the energy spectrum of the primary particles has an upper limit. This case, of considerable interest in work on accelerators, has been discussed by G.I. Kopylov [11]. Let us examine this case in detail.

From the foregoing, it is clear that for an analysis of the properties of a limited energy spectrum, it is necessary to find a value for the velocity V_1 of the primary particle, such that for any $V<V_1$ the value of E_κ should be greater than the left hand limit of the energy interval (12.13). In order to solve this problem, we shall investigate the behaviour of the function $\rho(V)=\gamma(\widetilde{E}_\kappa-\widetilde{p}_\kappa V)$ for various values of V.

If $V\to 0$, then $\frac{\partial\rho}{\partial V}<0$ and consequently $\rho(V)$ is a decreasing function. Therefore, for sufficiently small V, $\rho(V)<\widetilde{E}_\kappa$. As we have already observed, for sufficiently large V, $\rho(V)>\widetilde{E}_\kappa$. Therefore. the general trend of the function $\rho(V)$ is such that it decreases initially, passes through a certain minimum and then increases constantly, attaining for a certain V the value $\widetilde{E}_\kappa$. It is clear that within the limits $0<V<V^*$, the value of E_κ will always be found inside the energy interval of (12.13). Thus, the unknown $V_1=V^*$. Let us find V^* from the equation

$$\gamma_1^*(\widetilde{E}_\kappa-\widetilde{p}_\kappa V^*)=\widetilde{E}_\kappa. \quad (12.16)$$

This equation, besides having a trivial solution $V^*=0$, also has a second solution:

$$V^*=\frac{2\widetilde{p}_\kappa\widetilde{E}_\kappa}{\widetilde{E}_\kappa^2+\widetilde{p}_\kappa^2}. \quad (12.17)$$

Consequently, if the energy of the primary particles

$$E_1<\frac{M(\widetilde{E}_\kappa^2+\widetilde{p}_\kappa^2)}{m_1}, \quad (12.18)$$

then the spectrum of the secondary particles with mass m_1 will always have a characteristic maximum. From (12.18), it follows that: if $m_1\to 0$, then $E_1\to\infty$; in the converse case, $m_1\approx M$, $E_{1\,max}\approx M$. Having determined the values of $\widetilde{E}_{\kappa_1}$ and $\widetilde{E}_{\kappa_2}$ for a maximum in the spectrum the value of M can also be calculated.

We shall now consider the reduction of the secondary (primary) energy spectrum with respect to the state of the primary (secondary) spectrum. The general formulae quoted earlier (see Section 7) permit a complete solution for decay into two particles. However, the solution sometimes requires extensive calculations, and therefore we shall only consider a few problems.

First of all we shall consider the simplest case of the structure of the spectrum of the secondary particles.

Let, for example, π^0 -mesons be formed as a result of the reaction

$$\gamma + p \rightarrow \pi^0 + p$$

and let the angular distribution of the π^0 -mesons in the CM-system have the form

$$\widetilde{N}_{\pi^0}(\widetilde{\vartheta}) = A + B\cos^2\widetilde{\vartheta}\ ^*). \qquad (12.19)$$

In this case, the angular and energy distribution of the secondary photons, formed as a result of disintegration of the π^0 -mesons, has, in the CM-system, the form [12]:

$$N_\gamma(E, \vartheta) = \frac{1}{2q\gamma(1 - V\cos\vartheta)}\left\{A + \frac{B\sin^2\vartheta}{2\gamma^2(1 - V\cos\vartheta)^2} + \right.$$
$$\left. + \frac{3}{2}B\left[\frac{1}{\delta}\frac{m_\pi^2}{2q\gamma(1 - V\cos\vartheta)E}\right]^2\left[\left(\frac{\cos\vartheta - V}{1 - V\cos\vartheta}\right)^2 - \frac{1}{3}\right]\right\}, \qquad (12.20)$$

where

$$q = \sqrt{\bar{\varepsilon}^2 - m_\pi^2}, \quad \bar{\varepsilon} = \frac{2m_pE_\gamma + m_\pi^2}{2\sqrt{2m_pE_\gamma + m_p^2}}, \quad \delta = \frac{q}{\bar{\varepsilon}},$$

$V = \frac{E_\gamma}{E_\gamma + m_p}$, E_γ is the energy of the incident photon, E is the energy of the secondary photons. The angular distribution of the π^0 -mesons, represented by the function (12.19) has one more property** [13]. The energy spectrum

*The determination, in the general case for energy and angular distribution of the secondary particles, if the primary particles are moving along one direction, was quoted earlier (12.8), (12.10), (12.12).

**Irrespective of the method of production of the π^0 -mesons.

of the secondary photons, at angle $\vartheta_n = \arccos\frac{1}{\sqrt{3}}$, is symmetrical about the value of the energy $E_\gamma = \frac{m_\pi}{2}$, whether the π^0 -mesons are moving in one direction or even if their angular distribution is isotropic. Thus, the relationship (12.15) is applicable to the photons produced in the decay of π^0 -mesons having an angular distribution (12.19)*.

The problem of the reduction of the energy spectrum of primary particles decaying into a two-particles with respect to the spectrum of the secondary particles, has been discussed in works [11-16]. For a sufficiently large energy, namely when the condition $\operatorname{arch} E \gg \operatorname{arch}\widetilde{E}_\kappa$ is achieved, the solution was obtained in the form of a series [11]:

$$N(E_1) = -\frac{2\widetilde{p}_\kappa}{M}\sum_\Lambda(\varepsilon_\Lambda p_1 + \rho_\Lambda E_1)N_1'(\varepsilon_\Lambda E + \rho_\Lambda p), \qquad (12.21)$$

where

$$\varepsilon_\Lambda = \frac{m_1}{M}\operatorname{ch}\Lambda\operatorname{arch}\widetilde{E}_{\kappa_1},$$
$$\rho_\Lambda = \frac{m_1}{M}\operatorname{sh}\Lambda\operatorname{arch}\widetilde{E}_{\kappa_1},$$
$$(\Lambda = 1,\ 3,\ 5\ldots);$$

N_1' is derived from the energy spectrum of the particles with mass m_1. When $E_1 \approx p_1$, the expression (12.21) is somewhat simplified:

$$N(E_1) = -\frac{2m_1\widetilde{p}_\kappa}{M^2}E\sum_\Lambda e^{\Lambda\operatorname{arch}\widetilde{E}_{\kappa_1}}N_1'(Ee^{\Lambda\operatorname{arch}\widetilde{E}_{\kappa_1}}). \qquad (12.22)$$

In particular [16], for the decay $\theta^0 \rightarrow 2\pi$:

$$N(E_1) = -0{,}83E[0{,}91N'(0{,}91E) + 9{,}7N'(9{,}7E) + \ldots]. \qquad (12.23)$$

*On the properties of the angle ϑ_N, see also [14, 15].

CHAPTER IV

INTERACTIONS INVOLVING THREE SECONDARY PARTICLES

Reactions, in which three or more particles are present in the final state, are distinguished fundamentally from two-particle reactions by the number of parameters necessary to characterize them. Whilst reactions with two particles are characterized by two parameters, reactions with many particles naturally have a larger number of parameters and, in particular, there is no fixed value for the momentum $\tilde{p}_K$, which substantially determined the conclusions drawn in the preceding chapter. Consequently, for reactions involving many secondary particles, it is possible to pose only limited problems, not dependent upon the role of kinematic factors. In this case there exist three fundamental classes of problem: 1) Calculation of limiting relationships; 2) Calculation of the energy distributions arising from representations concerning the special role of phase factors; 3) Calculation of general tendencies in the extreme relativistic case, when it is possible, to a greater or lesser degree, to apply formula (4.3), into which the momentum distribution of the secondary particles does not enter.

Section 13. Limiting Relationships

If, as a result of some interaction, there are three secondary particles, then the Laws of Conservation of Energy and Momentum are written in the following form:

$$E_T = \sqrt{m_1^2 + p_1^2} + \sqrt{m_2^2 + p_2^2} + \sqrt{m_3^2 + p_3^2}. \qquad (13.1)$$

$$\mathbf{p}_T = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3. \qquad (13.2)$$

The trajectories of these particles no longer lie within one plane in the general case, and the relationship between the

angles of emission of the secondaries, or between the energy and angle of emission of any one of them is not definite. It is easy to show that if there are three secondary particles then even in CM-system, where the vectors of the momenta of the particles lie in one place, each one of the vectors is no longer characterized by a specific value for $\tilde{p}_K = f(\tilde{E}_T)$ (in accordance with 8.24), but any value from zero to a certain $\tilde{p}_{max}$ can be found in any direction. For this maximum, the momentum of the given particle (e.g. 1) corresponds to the emission of the other two particles as a single entity in the opposite direction. It is obvious that $\tilde{p}_{1\,max}$ and $\tilde{E}_{1\,max}$ are characterized in the given case by formulae (8.24) and (8.25), with the substitution of m_2 by $(m_2 + m_3)$. Thus, as a result of the formation of three particles, the value of the minor semi-axis of the momentum ellipse which appeared in its original construction, loses its distinct significance, and it is possible to speak only about its maximum value. The presence of only the upper limit (but not the lower, equal to zero) for the momentum of each of the particles in the CM-system by no means implies, of course, that in the L-system any momenta are also possible - up to a certain maximum - for any angle of emission. In each particular case, the possible intervals of the angles of emission of the particles and their momenta in the L-system, are determined by the relationship between the parameters of the ellipse, b and V, in accordance with the principle established above. One should note in addition, that the actual mechanism of the processes which result in three secondary particles may allow a number of more specific conclusions to be drawn - the mechanism involving two succesive acts with emission in each of them of two particles, is for example $\mathrm{I} + \mathrm{II} \rightarrow 1 + (2, 3) \rightarrow 2 + 3$. As a result of such a mechanism, the momentum in the CM-system of one of the three particles (in the given case, 1) is a maximum, and therefore a definite relationship is maintained between its direction of motion and the energy in the L-system, from the presence of such a relationship it is possible to draw a conclusion about the occurrence of an intermediate state ($(2, 3)$) during the course of the interaction being discussed.

In the very general case for reversion to a simple equation of the type (8.1-8.3), it is necessary to fix the momentum of one of the three particles, for example 3, and hence for $E' = E_T - \sqrt{m_3^2 + p_3^2}$ and $\mathbf{p}' = \mathbf{p}_T - \mathbf{p}_3$ they will be correct for all the relationships given above, and it is possible to plot the momentum ellipse for p_1 and p_2 for a fixed p_3. As a result of this, of course, the CM-system is no longer the centre of mass of the entire system, but only the centre of mass of particles *1* and *2*.

Let us consider a particular example of decay into three particles, namely β-decay of a nucleus M at rest $(p_T=0)$, and we shall look for possible values for the momenta of the electron (p_e) and the neutrino (p_ν), for a given momentum of the recoil nucleus p_R (which can vary from $p_{R\min}=0$ up to a certain $p_{R\max}$)). It is obvious that here $p'=-p_R$ and $E'=\Delta-W_R$, where $\Delta=M-m_R$ and m_R and W_R are the masses and kinetic energies of the recoil nuclei. In the centre of mass system of the electron and the neutrino, the total energy of these two particles $\tilde{E}'=\sqrt{(E')^2-(p')^2}=$ $=\sqrt{\Delta^2-2MW_R}$, and the minor semi-axis of the ellipse, in accordance with (8.24) is

$$b=\frac{\Delta^2-m_e^2-2MW_R}{2\sqrt{\Delta^2-2MW_R}}. \qquad (13.3)$$

Since $b>0$, it is obvious that $W_{R\max}=\frac{\Delta^2-m_e^2}{2M}$, and therefore the magnitude of the difference of energy of the initial and final nucleus Δ can be determined by the maximum energy of the recoil nucleus. For $W_R=W_{R\max}$, $b=0$, and also $a_\nu=f=0$, so that the neutrino receives none of the energy. Since $p_R^2\approx 2MW_R$, then (13.3) can be written in the form

$$b=\frac{p_{R\max}^2-p_R^2}{2\sqrt{\Delta^2-p_R^2}}. \qquad (13.4)$$

The recoil nucleus removes only a small portion of the energy of the β-disintegration and consequently the eccentricity of the momentum ellipse

$$V=\frac{p'}{E'}\approx\frac{p_R}{\Delta}, \qquad (13.5)$$

and its major semi-axis

$$a\approx\frac{b\Delta}{\sqrt{\Delta^2-p_R^2}}=\Delta\,\frac{p_{R\max}^2-p_R^2}{2(\Delta^2-p_R^2)}. \qquad (13.6)$$

For $p_R=0$, the momentum ellipse is turned into a circle, and for $p_{R\max}$ into a point. With increase of p_R, in this manner both values a and b decrease. For a definite value of α_n we easily obtain, in accordance with (10.7)

$$\alpha_e=\frac{p_R}{\sqrt{\Delta^2-p_R^2}}\sqrt{m_e^2+\frac{p_{R\max}^2-p_R^2}{4(\Delta^2-p_R^2)}}. \qquad (13.7)$$

Since

$$p_{R\max}^2=\Delta^2-m_e^2, \qquad (13.8)$$

then

$$\alpha_e=\frac{p_R}{2}\left[1+\frac{m_e^2}{\Delta^2-p_R^2}\right]=f\,\frac{(\Delta^2-p_R^2)+m_e^2}{(\Delta^2-p_R^2)-m_e^2}. \qquad (13.9)$$

In a similar manner, for the neutrino we have

$$\alpha_\nu=f=\frac{p_R}{2}\left[1-\frac{m_e^2}{\Delta^2-p_R^2}\right]. \qquad (13.10)$$

The upper and lower limits of the kinetic energy of the electron for a given momentum of the recoil nucleus are obtained from (10.12) and (10.13) in the form:

$$W_{e\min}=\frac{(\Delta-p_R-m_e)^2}{2(\Delta-p_R)} \qquad (13.11)$$

and

$$W_{e\max}=\frac{(\Delta+p_R-m_e)^2}{2(\Delta+p_R)}. \qquad (13.12)$$

The discussion presented is a typical example of analysis of the kinematics of interactions involving three secondary particles via reduction to various particular cases of paired combinations of two particles. In the example given, we have obtained in this way, the maximum energy of the recoil nucleus and the limits for the kinetic energy of the β-particle for a given energy of the nucleus. Ultimately, we shall return to analysis of interactions involving three and more secondary particles, but for the present we shall pass on to the determination of the limiting angles.

From formula (7.14) it follows that the maximum angle, through which a particle with a mass m_1 is deflected, increases with its momentum in the CM-system. Therefore, the maximum permissible angle $\vartheta_{\max}$ is determined by formula (11.7), where in place of $\tilde{p}_x$ it is necessary to substitute the value $\tilde{p}_{\max}$.

If the mass of one of the particles is negligibly small compared with the mass of the other secondary particles, then $\widetilde{p}_{max}$ is converted to $\widetilde{p}_{K}$. Consequently, the method of analysis described in Section 11 does not discriminate very well between events involving a decay into two particles, or decay into three particles, of which at least one has a mass negligibly small compared with the masses of the other particles.

It is well-known that it is also necessary to be cautious in the analysis of the permissible angles ψ (see Section 11).

A similar ambiguity can arise in the analysis of the complanarity of the tracks of the primary and the two charged secondary particles (see Section 11).

Actually, it is possible to demonstrate the need for care by the following case. Let the particle decay into two charged and one neutral particles. Hence, if the mass of one of the two secondary charged particles (e.g. m_2) is close to the mass of the primary particle M and the velocity of the latter $V \approx 1$, then the tracks of both the charged and primary particles are almost co-planar. In the L-system relationship (13.2) is fulfilled.

Let us multiply both parts of equation (13.2) scalarly by the single vector $\boldsymbol{n}$, normal to the vectors $\boldsymbol{p}_I (\boldsymbol{p}_I = \boldsymbol{p}_T)$ and $\boldsymbol{p}_1$; we obtain

$$(\boldsymbol{n}\boldsymbol{p}_2) = -(\boldsymbol{n}\boldsymbol{p}_3). \qquad (13.13)$$

Since in the given case

$$p_2 \approx \gamma V \widetilde{E}_2 \gg \gamma (\widetilde{p}_3 + V \widetilde{E}_3) \approx p_3,$$

then

$$|\cos\vartheta_2| = \frac{p_3}{p_2} |\cos\vartheta_3| \ll 1 \qquad (13.14)$$

(ϑ_2 and ϑ_3, are respectively the angles between the vectors $\boldsymbol{p}_2$ and $\boldsymbol{n}$ and $\boldsymbol{p}_3$, and $\boldsymbol{n}$); consequently, the vector $\boldsymbol{n}$ is close to the normal to the momentum $\boldsymbol{p}_2$. Thus, if the $\boldsymbol{n}$ particle disintegrates into two charged (whereupon $m_2 \approx M$) and one neutral particles, then the vectors $\boldsymbol{p}_I$, $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ are almost co-planar.

Section 14. Energy Spectrum of the Secondary Particles

For a rigorous calculation of the energy spectrum of the secondary particles, formed as a result of a decay into three particles, it is necessary to know the mode of interaction of the particles with a field. However, up to the present, in not one case has it been possible to establish the nature of such interactions. Therefore, it is advisable, in order to obtain an estimate of the energy spectrum, to use approximate methods. The accuracy of the usual methods of evaluation of the characteristics of decay processes depends on the assumption that they are determined only by the statistical importance of the final states and do not depend on the mode of interaction. It should therefore be expected that such an approach is justified in every case for weak interactions. Thus, the evaluation of the form of the energy spectra of the electrons, formed as a result of the decay of μ-mesons and by β-decay, are approximately in accordance with experimental data [17]. A similar method was used with success to study multiple processes for high energies of the interacting particles [18] (See Chapter V).

Evaluations of energy spectra as a result of such a supposition were carried out in works [19, 20]. It has been shown that in the CM-system, the probability dw that a particle with mass m_1 will have a momentum between $\widetilde{p}_1$, $\widetilde{p}_1 + d\widetilde{p}_1$, is determined by the expression:

$$dw \infty \frac{B^{1/2}\widetilde{p}_1^2}{A^2 - \widetilde{p}_1^2} \left[\left(1 - \frac{4A^2}{A^2 - \widetilde{p}_1^2}\right) B + AC \right] d\widetilde{p}_1, \qquad (14.1)$$

where

$$A = M - \sqrt{\widetilde{p}_1^2 + m_1^2},$$
$$B = (\widetilde{p}_1^2 + m_2^2 + m_3^2 - A^2)^2 - 4m_2^2 m_3^2,$$
$$C = 6A[A^2 - (\widetilde{p}_1^2 + m_2^2 + m_3^2)].$$

Naturally, the distribution is determined only by the initial masses. In order to proceed to the energy representation, it is necessary to substitute $\widetilde{p}_1$ in (14.1) by

$$\sqrt{\widetilde{E}_1^2 - m_1^2} \text{ and } d\widetilde{p}_1 \text{ by } \frac{\widetilde{E}_1}{\widetilde{p}_1} d\widetilde{E}_1.$$

Let us consider particular cases of the relationship (14.1)

1)

$$m_i \ll \tilde{p}_i \qquad (i = 1,\ 2,\ 3).$$

$$dw_1 \propto \left(3M^2 - 6M\tilde{p}_1 + 2\tilde{p}_1^2\right)\tilde{p}_1^2\, d\tilde{p}_1; \tag{14.2}$$

2)

$$m_i \gg \tilde{p}_i,$$

$$dw_1 \propto \left[2(m_2 + m_3)\tilde{W}_\kappa - \frac{M}{m_1}\tilde{p}_2^2\right]^{1/2}\tilde{p}_1^2\, d\tilde{p}_1, \tag{14.3}$$

3)

$$m_1 \ll \tilde{p}_1,\quad m_2 \gg \tilde{p}_2,\quad m_3 \gg \tilde{p}_3,$$

$$dw_1 \propto (\tilde{W}_\kappa - \tilde{p}_1)^{1/2}\tilde{p}_1^2\, d\tilde{p}_1; \tag{14.4}$$

4)

$$m_1 \gg \tilde{p}_1,\quad m_2 \ll \tilde{p}_2,\quad m_3 \gg \tilde{p}_3,$$

$$dw_1 \propto \left(\tilde{W}_\kappa^2 + \tilde{p}_1^2\right)\tilde{p}_1^2\, d\tilde{p}_1, \tag{14.5}$$

5)

$$m_1 \ll \tilde{p}_1,\quad m_2 \ll \tilde{p}_2,\quad m_3 \gg \tilde{p}_3.$$

$$dw_1 \propto \left(\tilde{W}_\kappa^2 - \tilde{p}_1^2\right)\tilde{p}_1^2\, d\tilde{p}_1. \tag{14.6}$$

We shall calculate the energy and angular distributions of the secondary particles for decay in flight. These calculations also allow us to draw some fairly general conclusions for the relativistic case.

Let us calculate the energy and angular distribution in the L-system if the distribution in the CM-system corresponds to function (14.2).

In this case, in the CM-system

$$\tilde{N}(\tilde{p}_1)\, d\tilde{p}_1\, d\cos\tilde{\vartheta} \propto \left(3M^2 - 6M\tilde{p}_1 + 2\tilde{p}_1^2\right)\tilde{p}_1^2\, d\tilde{p}_1\, d\cos\tilde{\vartheta}. \tag{14.7}$$

For the functions $J(\vartheta)$ and $\tilde{p}(p)$ we shall use the approximate expressions (7.23) and $\tilde{p} = \gamma E(1 - V\cos\vartheta)$. Hence, the momentum distribution

$$N(p_1)\, dp_1 \propto dp_1 \int_{\cos\vartheta_{max}(p)}^{1} \frac{\left(3M^2 - 6M\tilde{p}_1 + 2\tilde{p}_1^2\right)\tilde{p}_1^2}{1 - V\cos\vartheta}\sin\vartheta\, d\vartheta =$$

$$= dp_1 \int_{\cos\vartheta_{max}(p)}^{1} [3M^2 - 6Mp_1(1 - V\cos\vartheta) +$$

$$+ 2p_1^2(1 - V\cos\vartheta)^2]\, p_1^2(1 - V\cos\vartheta)\sin\vartheta\, d\vartheta. \tag{14.8}$$

Hence it follows that

$$\cos\vartheta_{max} = \begin{cases} \frac{1}{V}\left(1 - \frac{\tilde{p}_{1max}}{\gamma p_1}\right) & \text{for } \frac{1}{V}\left(1 - \frac{\tilde{p}_{1max}}{\gamma p_1}\right) > -1, \\ -1 & \text{for } \frac{1}{V}\left(1 - \frac{\tilde{p}_{1max}}{\gamma p_1}\right) < -1. \end{cases} \tag{14.9}$$

After integration we obtain

$$N(p_1)\, dp_1 \propto p_1^2 \{3M^2[\chi^2 - (1 - V)^2] -$$

$$- 4\gamma M p_1[\chi^3 - (1 - V)^3] + \gamma^2 p_1^2[\chi^4 - (1 - V)^4]\}; \tag{14.10}$$

$$\chi = \begin{cases} \frac{M}{2p_1\gamma} & \text{for } p_1 > \frac{M}{2\gamma(1 + V)}, \\ 1 + V & \text{for } p_1 < \frac{M}{2\gamma(1 + V)}. \end{cases}$$

The values of the momentum p_1 are included within the interval from 0 to $\gamma\frac{M}{2}(1+V)$.

In the case of interest to us (relativistic particles), a very general relationship can be established for average momenta. In accordance with (12.10), the average momentum $\bar{p}_1$ in the L-system, corresponding to a fixed momentum $\tilde{p}_1$, is equal to: $\bar{p}_1 \approx \gamma\tilde{p}_1$, whence in the case of an arbitrary distribution in the CM-system

$$\bar{p}_1 \approx \gamma \int_0^{\frac{M}{2}} \tilde{N}(\tilde{p}_1)\tilde{p}_1\, d\tilde{p}_1 = \gamma\bar{\tilde{p}}_1. \tag{14.11}$$

Proceeding to calculation of the angular distribution

$$N(\vartheta)\, d\cos\vartheta \propto d\cos\vartheta \int_0^{p_{max}(\vartheta)} [3M^2 - 6Mp_1(1 - V\cos\vartheta) + \tag{14.12}$$

$$+ 2p_1^2(1 - V\cos\vartheta)^2]\, p_1^2(1 - V\cos\vartheta)\, dp_1.$$

whereupon

$$p_{max}(\vartheta) = \frac{M}{2\gamma(1 - V\cos\vartheta)}.$$

After integration we obtain

$$N(\vartheta)\, d\cos\vartheta \infty \frac{d\cos\vartheta}{(1 - V\cos\vartheta)^2}. \tag{14.13}$$

This expression agrees, if it be normalized, with the angular distribution of ultra-relativistic particles, arising as a result of decays into two particles (12.9). Such agreement is by no means a mere chance. The special feature of the expression is its independence of the energy of the secondary particles (provided that they have sufficiently large velocities). Therefore, for any momentum distribution of ultra-relativistic particles in the CM-system, their angular distribution in the L-system will be determined by relationship (12.3).

CHAPTER V

MULTIPLE PROCESSES

For increased energies of colliding particles, the possibility of the formation of many particles in one act of collision is increased. Of particular interest are multiple processes with participation of mesons. In this chapter we shall consider the kinematics of such processes. To them, to an even greater extent than in the case with three secondary particles, the comment concerning the impossibility of a purely kinematic description of the processes applies. Consequently, particular problems will be treated in the same way as in the previous chapter.

Section 15. Limiting Relations

In the general case, two overall relationships can be obtained: for the maximum momentum of a given particle with a mass m_1 in the CM-system and for the maximum angle of emission of this particle in the L-system. It is obvious that the second relationship is determined, to a considerable degree, by the first.

Determination of Limiting Momenta in the CM-system

The momentum of a certain particle will have a maximum value if all the remaining particles 2, 3, ... N are moving in the opposite direction.

This circumstance substantially simplifies the ensuing discussion, in so far as in place of vector dimensions, it is possible to consider scalar dimensions.

Since the energy and momentum of particle I have the following values:

$$\widetilde{E}_1 = \widetilde{E}_1 + m_{II} - \sum_{i=2}^{N} \widetilde{E}_i = \widetilde{E}_T - \sum_{i=2}^{N} E_i,$$

$$\widetilde{p}_1 = \sum_{i=2}^{N} \widetilde{p}_i,$$

then its velocity

$$\widetilde{\beta}_1 = \frac{\widetilde{p}_1}{\widetilde{E}_1} = \frac{\sum_{i=2}^{N} \widetilde{p}_i}{\widetilde{E}_T - \sum_{i=2}^{N} \widetilde{E}_i}. \quad (15.1)$$

We require to find the maximum value of this quantity for the condition

$$\sum_{i=1}^{N} \widetilde{E}_i = \widetilde{E}_T, \quad (15.2)$$

i.e. to solve the problem by calculation of the limiting condition. Following Lagrange's method, we shall look for the absolute limit of the expression

$$K = \widetilde{\beta}_1 + Q \sum_{i=1}^{N} \widetilde{E}_i, \quad (15.3)$$

where Q is a factor which will be determined later on. The limiting condition K is determined by the system equation

$$\frac{\partial K}{\partial \widetilde{p}_j} = 0, \quad j = 2 \ldots N. \quad (15.4)$$

Since $\frac{\partial \widetilde{E}_j}{\partial \widetilde{p}_j} = \widetilde{\beta}_j$, then this system can be written in the form

$$\frac{1}{\widetilde{E}_T - \sum_{i=2}^{N} \widetilde{E}_i} + \frac{\sum_{i=2}^{N} \widetilde{\beta}_j \widetilde{p}_i}{\left(\widetilde{E}_T - \sum_{i=2}^{N} \widetilde{E}_i\right)^2} + Q \frac{\sum_{i=2}^{N} \widetilde{p}_i}{\widetilde{E}_T - \sum_{i=2}^{N} \widetilde{E}_i} = 0, \quad (15.5)$$

$$j = 2 \ldots N.$$

Here the value of Q should satisfy all the $N-1$ equations. This is possible only if all the equations are identical, i.e. if it fulfills the condition

$$\widetilde{\beta}_2 = \widetilde{\beta}_3 = \ldots = \widetilde{\beta}_N. \quad (15.6)$$

The method presented for the proof of relationship (15.6) is due to Sternheimer [21]. It follows that for the limiting case, particles $2 \ldots N$ are equivalent to one particle with a mass $\sum_{i=2}^{N} m_i$.

Hence, the determination of the maximum momentum of the particle reduces to the problem of two particles. Consequently, by formula (10.55),

$$E_{1\,max} = \frac{\widetilde{E}_T^2 + m_1^2 - \left(\sum_{i=2}^{N} m_i\right)^2}{2\widetilde{E}_T} \quad (15.7)$$

and

$$p_{1\,max} = \frac{\sqrt{\widetilde{E}_T^4 + m_1^4 + \left(\sum_{i=2}^{N} m_i\right)^4 - 2\left\{\widetilde{E}_T^2 m_1^2 + \widetilde{E}_T^2 \left(\sum_{i=2}^{N} m_i\right)^2 + m_1^2 \left(\sum_{i=2}^{N} m_i\right)^2\right\}}}{2\widetilde{E}_T}. \quad (15.8)$$

In the more general form this problem is formulated for the case where the momenta of the primary particles 1, 2, ..., d are given. Hence, the maximum momentum of the $(d+1)$-th particle is oppositely directed to the summed momenta: $\widetilde{\boldsymbol{p}}_\Sigma = \sum_{i=1}^{d} \widetilde{\boldsymbol{p}}_i$, and its value is determined from the relationships

$$\widetilde{p}_{d+1} = \widetilde{p}_\Sigma + \sum_{i=d+2}^{N} \widetilde{p}_i, \quad (15.9)$$

$$\sum_{i=1}^{N} \widetilde{E}_i = \widetilde{E}_T. \quad (15.10)$$

Proceeding to the calculation of the angles ϑ_{max} in the L-system we note, first of all, that ϑ_{max} is increased for increase of $\widetilde{p}$ (excluding the case when this angle attains its limiting value $\frac{\pi}{2}$). Therefore the limiting angle corresponds to $\widetilde{p}_{max}$ and is determined by relationship (7.14).

Substituting the value for the velocity $\widetilde{\beta}_1$ from equation (15.1), we have finally

$$\vartheta_{max} = \arcsin \sqrt{\frac{(1 - V^2) A}{4\widetilde{E}_T^2 m_1^2 V^2}}, \quad (15.11)$$

where

$$A=\left[\left(\widetilde{E}_T+\sum_{i=2}^{N}m_i\right)^2-m_1^2\right]\left[\left(E_T-\sum_{i=2}^{N}m_i\right)^2-m_1^2\right]. \quad (15.12)$$

Figures 23-27 show the maximum angles for πN - and NN - collisions for various energies.

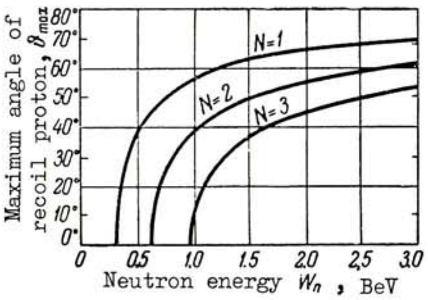


Fig. 23

Dependence of maximum angle ϑ_{max} of the recoil proton in np-collisions with formation of N π-mesons, on the kinetic energy W_n of the incident neutron in the L-system [21]

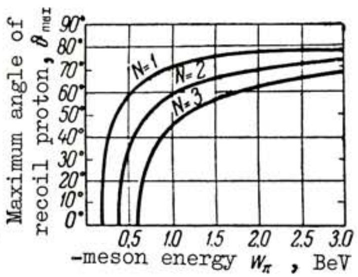


Fig. 24

Dependence of maximum angle ϑ_{max} of the recoil proton in πp-collisions with formation of N π-mesons, on the kinetic energy W_π of the incident π-meson [21]

For the derivation of (7.14) and consequently also (15.11) it has been assumed that in the CM-system the particles can be emitted at any angle; however, in principle the case is possible (it is brought about as a result of collision of particles of sufficiently large energies E_T) when the angular distribution of the secondary particles in the CM-system is essentially anisotropic, and consequently, in the CM-system there is also a certain limiting angle $\widetilde{\vartheta}_{extr}$. We shall derive the corresponding angle in the L-system, assuming that in the CM-system the dispersion takes place symmetrically relative to a plane perpendicular to the direction of motion of the particles. In this case corresponds to the particle moving through a limiting angle in the rear hemisphere in the CM-system

$$\vartheta'_{max}=\mathrm{arctg}\,\frac{1}{\gamma}\,\frac{\sin\widetilde{\vartheta}_{extr}}{\cos\widetilde{\vartheta}_{extr}+\dfrac{V}{\sqrt{\dfrac{A}{A+4\widetilde{E}_T^2 m_1^2}}}}, \quad (15.13)$$

if ϑ'_{max} is the largest angle ϑ_{max}, determined by (15.11). In the alternative case it is necessary to use (15.11).

Another problem arises in that case, if the momenta of the

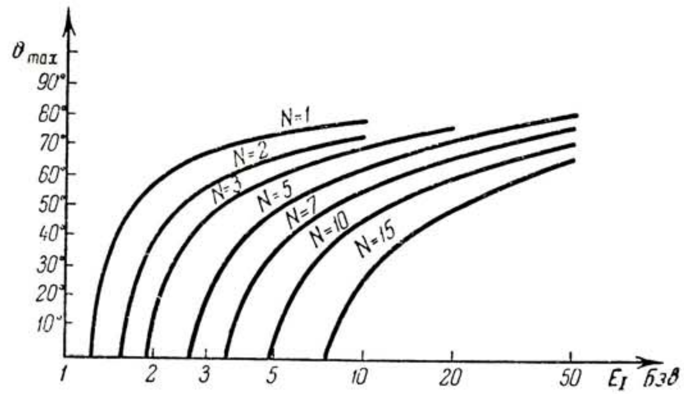


Fig. 25

Dependence of the maximum angle ϑ_{max} of the recoil nucleus in nucleon-nucleon collisions of high energies, with formation of N π-mesons, upon the total energy of the incident nucleon. Curves calculated by N.G. Birger.

many secondary particles are measured. Let us consider the case when the momenta are known for all the secondary particles except one (1), for which it is required to determine the maximum angle [22].

The equation of conservation for this case can be written in the form:

$$[E_I-p_I]+m_{II}=\sum_{i=2}^{N}(E_i-p_i\cos\vartheta_i)+E_1-p_1\cos\vartheta_1. \quad (15.14)$$

If $m_{II}\neq 0$ and the energy E_1 is sufficiently large, then

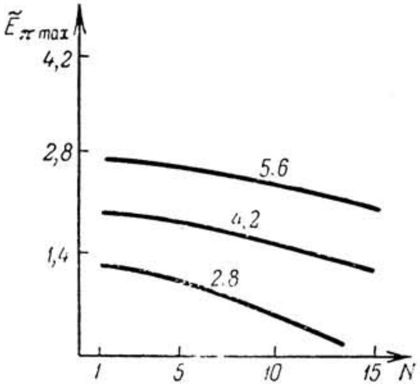


Fig. 26

Maximum energy $\tilde{E}_{\pi\,\max}$ of the π-meson formed by meson - nucleon collisions with formation of $N\pi$ -mesons [33]. The energy is measured in BeV. Numbers by the curves indicate energy of the primary π-mesons in the CM-system

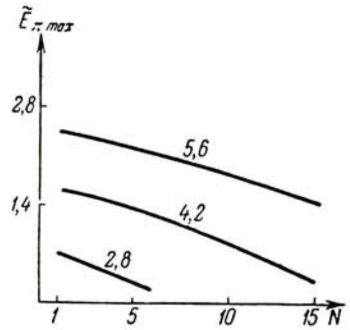


Fig. 27

Maximum energy $\tilde{E}_{\pi\,\max}$ of the π-meson formed by nucleon - nucleon collisions with formation of $N\pi$ -mesons [33]. The energy is measured in BeV. Numbers by the curves indicate the energy of the primary π-mesons in the CM-system

the term $E_1 - p_1 \ll m_{II}$, and can be neglected. Then

$$p_1 = \frac{1}{\sin^2\vartheta_1}\left[B\cos\vartheta_1 \pm \sqrt{B^2 - m_1^2\sin^2\vartheta_1}\right]. \tag{15.15}$$

where

$$B = m_{II} - \sum_{i=2}^{N}(E_i - p_i\cos\vartheta_i) \tag{15.16}$$

and consequently

$$\vartheta_{1\,\max} = \arcsin\frac{B}{m_1}. \tag{15.17}$$

As usual, if $\frac{B}{m_1} \geqslant 1$, then $\vartheta_{1\,\max} = \frac{\pi}{2}$. We shall consider finally the determination of the thresholds of formation of new particles in collision processes. The minimum energy

necessary for formation of new particles corresponds to the case, when all the particles after the reaction are at rest in the CM-system. This corresponds to the condition:

$$\sqrt{m_I^2 + \tilde{p}^2} + \sqrt{m_{II}^2 + \tilde{p}^2} = \sum_{i=1}^{N} m_i, \tag{15.18}$$

where $\tilde{p}$ is the momentum of the colliding particles. Hence it follows

$$\tilde{p} = \sqrt{\left[\frac{\left(\sum_{i=1}^{N} m_i\right)^2 + m_I^2 - m_{II}^2}{2\sum_{i=1}^{N} m_i}\right]^2 - m_I^2}, \tag{15.19}$$

$$\tilde{E}_I = \frac{\left(\sum_{i=1}^{N} m_i\right)^2 + m_I^2 - m_{II}^2}{2\sum_{i=1}^{N} m_i}, \tag{15.20}$$

$$\tilde{E}_{II} = \frac{\left(\sum_{i=1}^{N} m_i\right)^2 + m_{II}^2 - m_I^2}{2\sum_{i=1}^{N} m_i} \tag{15.21}$$

and

$$E_{I\,\text{Thresh}} = \gamma(\tilde{E}_I + V\tilde{p}), \tag{15.22}$$

where

$$V = \frac{\tilde{p}}{\tilde{E}_{II}}. \tag{15.23}$$

Actually, if $m_I = m_{II} = m$, then

$$\tilde{E}_I = \tilde{E}_{II} = \frac{\sum_{i=1}^{N} m_i}{2} \quad \text{and} \quad \gamma = \frac{\sum_{i=1}^{N} m_i}{2m}$$

Section 16. Angular and Energy Distributions of Multiple Processes

It has already been mentioned that for multiple processes it is impossible, from a purely kinematic consideration, to obtain the characteristics of these processes. However, the idea advanced by Fermi that the characteristics of multiple processes leading to the formation of particles are determined by the statistical weight of the final states is in reasonable agreement with experiment. A certain similarity between these methods can be found in the fact that neither of them take into account the details of the interaction between the particles, although the very fact of establishing statistical equilibrium is a consequence of this interaction (and herein is ultimately the essential difference between the statistical and kinematic conception). A detailed review of statistical theory exceeds the bounds of this book. We shall only mention that in the energy region around several BeV this theory, taking into account the isobaric states of the nucleons, predicts correctly the energy distribution and the multiplicity of π-mesons formed* for energies of $\tilde{E}_n = 1$—6 BeV.

Formally the contents of the statistical theory can be written in the form of the simple relationships

$$W_N(\tilde{E}_T) \infty \rho_{\tilde{E}_T} \quad {**}) \qquad (16.1)$$

and

$$w_N(\tilde{E}_T, \tilde{p}_1)\,d\tilde{p}_1 = 4\pi\tilde{p}_1^2 w_{N-1}(\tilde{E}_T - \sqrt{m_1^2 + \tilde{p}_1^2}, \tilde{p}_1)\,d\tilde{p}_1 \, {***}), \qquad (16.2)$$

where W_N is the probability of formation of N-particles, if the total energy is equal to $\tilde{E}_T$, and the total momentum

*For a detailed survey of statistical theory see [23]. We note additionally that for very high energies ($E_T \gg 10$ BeV) the statistical theory, not taking into account the interaction between mesons, is not justified. For a description of this interaction L.D. Landau has suggested the use of relativistic hydrodynamics for an ideal liquid [24].

**In general, the processes are considered in the CM-system.

***We note that for calculation of the momentum distribution of the system for N-particles the function is considered

$$w_{N-1}(\tilde{E}_T - \sqrt{m_1^2 + \tilde{p}_1^2}, \tilde{p}_1),$$

where $m_1, \tilde{p}_1$ are the mass and momentus of a certain specified particles.

is equal to zero; $\rho_{\tilde{E}_T}$ is the density of states, and $w_N(\tilde{E}_T, \tilde{p}_1)\,d\tilde{p}_1$ is the probability that in the system under consideration, one of the particles has a momentum within the interval $\tilde{p}_1, \tilde{p}_1 + d\tilde{p}_1$. We shall write the expression for the density of states of the system with an energy $\tilde{E}_T$ and with an momentum $\tilde{p}_T$, in the form

$$\left(\frac{V}{8\pi^3}\right)^{N-1} \frac{dQ_N(\tilde{E}_T, \tilde{p}_n)}{d\tilde{E}_T} q_{N,T} \sim w_N(\tilde{E}_T, \tilde{p}_T), \qquad (16.3)$$

where V is a certain characteristic volume (for the case of collision of 2 nucleons this volume, in absolute value is equal to $\left(\frac{1}{\mu_\pi}\right)^3$); the exponent of the power is equal to $N-1$ (and not N) in consequence of the Law of Conservation of momentum, and $q_{N,T}$ is a factor which takes into account the conservation of the normal and isotropic spins; it is calculated according to well-known rules of quantum-mechanics (see [25] and Chapter VI). The factor $\frac{dQ_N(\tilde{E}_T, \tilde{p}_T)}{d\tilde{E}_T}$, equal to the volume occupied by the system in momentum space, for the conditions

$$\sum_{i=1}^{N} \tilde{E}_i = \tilde{E}_T, \qquad (16.4)$$

$$\sum_{i=1}^{N} \tilde{p}_i = \tilde{p}_T \qquad (16.5)$$

is equal to the $3N$-dimensional integral

$$\underbrace{\int \ldots \int}_{3N} \delta\left[\tilde{E}_T - \sum_{i=1}^{N} \sqrt{m_i^2 + \tilde{p}_i^2}\right] \delta\left[\tilde{p}_T - \sum_{i=1}^{N} \tilde{p}_i\right] \prod_{i=1}^{N} d^3\tilde{p}. \qquad (16.6)$$

In the general case, the evaluation of this integral is intricate; it was carried out recently in [26]. The result was obtained in the form of a complex non-power expansion. We shall confine ourselves here to the two extreme cases, useful in practice: the ultra-relativistic (all $m_i = 0$)

and the non-relativistic case $\left(\widetilde{E}_i = \frac{\widetilde{p}_i^2}{2m_i}\right)$ [27, 28].

In these cases

$$\frac{dQ_N(\widetilde{E}_T, \widetilde{p}_T)}{d\widetilde{E}_T} = \frac{(2\pi)^{\frac{3}{2}(N-1)}}{\left[\frac{3}{2}(N-1)-1\right]!}\left[\frac{m_1 m_2 \dots m_N}{\sum_{i=1}^{N} m_i}\right]^{\frac{3}{2}}\left[\widetilde{W}_T - \frac{\widetilde{p}_T^2}{2\sum_{i=1}^{N} m_i}\right]^{\left[\frac{3}{2}(N-1)-1\right]} \quad (16.7)$$

where W_T is the total kinetic energy of the particle (non-relativistic case) and

$$\frac{dQ_N(\widetilde{E}_T \widetilde{p}_T)}{d\widetilde{E}_T} = \frac{\pi^{N-1}}{2^{N-2}} \frac{(\widetilde{E}_T^2 - \widetilde{p}_T^2)^{N-2}}{\widetilde{p}_T} \times$$
$$\times \sum_{i=1}^{N} C_N^i \frac{(\widetilde{E}_T - \widetilde{p}_T)^i (\widetilde{E}_T + \widetilde{p}_T)^{N-i}}{(N+i-2)!\,(2N-i-2)!}\left[\frac{\widetilde{E}_T + \widetilde{p}_T}{2N-i-1} - \frac{\widetilde{E}_T - \widetilde{p}_T}{N+i-1}\right] \quad (16.8)$$

(ultra-relativistic case).

From (16.7) and (16.8) respectively, we obtain

$$\frac{dQ_N(\widetilde{E}_T, 0)}{d\widetilde{E}_T} = \frac{(2\pi)^{\frac{3}{2}(N-1)}}{\left[\frac{3}{2}(N-1)-1\right]!}\left(\frac{\prod_{i=1}^{N} m_i}{\sum_{i=1}^{N} m_i}\right)^{3/2} \widetilde{W}_T^{\frac{3}{2}(N-1)-1}, \quad (16.9)$$

$$\frac{dQ_N(\widetilde{E}_T, 0)}{d\widetilde{E}_T} = \left(\frac{\pi}{2}\right)^{N-1} E_T^{3N-4} \frac{(4N-4)!\,(2N-1)}{(3N-4)!\,[(2N-1)!]^2}. \quad (16.10)$$

Omitting unimportant multiples (which should be taken into account by normalization), one can write

$$w_N(\widetilde{E}_T, p_1) = p_1^2\left\{\widetilde{W}_T - \frac{\widetilde{p}_1^2}{2}\left[\frac{\sum_{i=1}^{N} m_i}{m_1 \sum_{i=2}^{N} m_i}\right]\right\} \quad (16.11)$$

and

$$w_N(\widetilde{E}_T, \widetilde{p}_1) = \widetilde{p}_1(\widetilde{E}_T - 2\widetilde{p}_1)^{N-3} \times$$
$$\times \sum_{i=0}^{N-1} \frac{C_{N-1}^i \widetilde{E}_T^{N-i-1} (\widetilde{E}_T - 2\widetilde{p}_1)^i}{(N+i-3)!\,(2N-i-4)!}\left[\frac{\widetilde{E}_T}{2N-i-3} - \frac{\widetilde{E}_T - 2p_1}{N+i-2}\right]. \quad (16.12)$$

Figure 28 shows the momentum distribution of π-mesons formed as a result of annihilation of anti-nucleons at rest; Figure 29 shows the relationship between the average multiplicity of π-mesons and the kinetic energy of the colliding nucleons. The momentum distributions in accordance with (16.2), (16.11) and (16.12) are given in the CM-system. In order to determine them in the L-system, it is necessary to know the angular distribution of these particles in the CM-system. Unfortunately, the theory in its present form does not permit accurate calculations of this distribution, since to do this it is necessary to take into consideration the Law of Conservation of angular Momentum for a number of motions, which has not hitherto been done. However, for comparatively moderate energies ($\lesssim 5$ BeV), experimental data shows that the angular distribution of the secondary particles in the CM-system is approximately isotropic. Assuming the distribution to be isotropic we can, for conversion from the CM-system to the L-system, apply formula (7.4) or (7.18) and then derive the integration with respect to momentum.

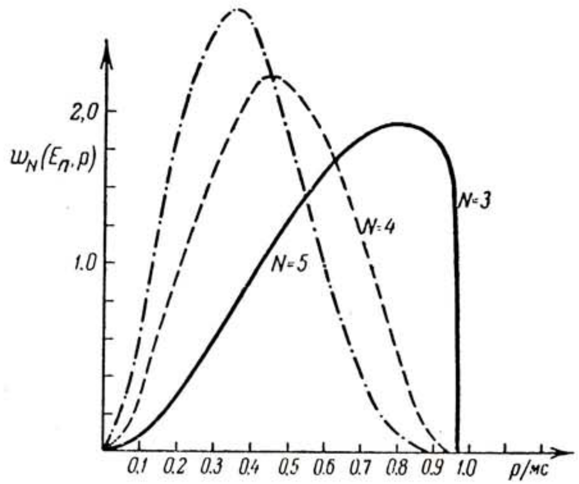


Fig. 28

Momentum distribution of π-mesons formed as a result of annihilation of stationary anti-nucleons. The distribution was calculated by V.M. Maksimenko [34] according to precise formulae from Statistical theory

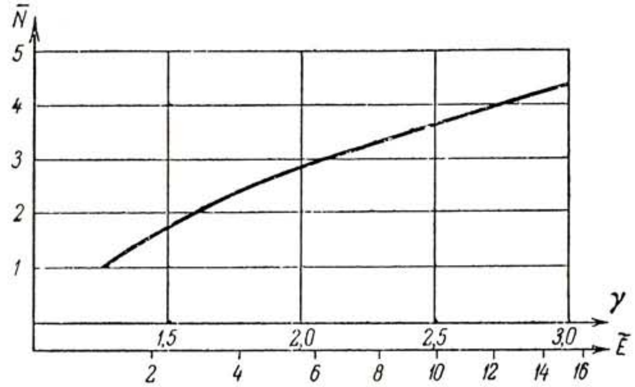


Fig. 29

Dependence of the average multiplicity of π-mesons formed as a result of nucleon-nucleon collision, on the kinetic energy of the participating nucleons. The calculations were carried out in accordance with formulae from statistical theory with the assumption of an isobaric state [23]

It may be expected that a similar approach to the solution of the problem will not hold good for very large energies when, in the CM-system, an appreciable departure from isotropy occurs (see for example [24]). Bearing in mind such a case, we shall derive the conversion formulae from the CM-system into the L-system, when the angular distribution in the first system is assumed a cosine function, and the momentum distribution a δ-function [5].

$$\widetilde{N}(\widetilde{\vartheta}, \widetilde{p}) \sin\widetilde{\vartheta}\, d\widetilde{\vartheta}\, dp = \frac{(2k+1)}{4\pi} \cos^{2k}\widetilde{\vartheta}\, \delta(\widetilde{p} - \widetilde{p}_1) \sin\widetilde{\vartheta}\, d\widetilde{\vartheta}\, d\widetilde{p}_1, \quad (16.13)$$

where $k = 1, 2, 3, \ldots$. By using (7.18) and integrating with respect to momentum, it is possible to obtain

$$N(\vartheta)\sin\vartheta\, d\vartheta = \frac{(2k+1)}{2} \frac{(\gamma \operatorname{tg}\vartheta)^2}{(\gamma^2 \operatorname{tg}^2\vartheta + 1)^{2k+2}\cos\vartheta} \times$$
$$\times \left\{ \frac{\left[\frac{\widetilde{\beta}}{V} + \sqrt{1 - \left[\left(\frac{\widetilde{\beta}}{V}\right)^2 - 1\right]\gamma^2 \operatorname{tg}^2\vartheta}\right]^2 \left[-\frac{\widetilde{\beta}}{V}\gamma^2 \operatorname{tg}^2\vartheta + \sqrt{1 - \left[\left(\frac{\widetilde{\beta}}{V}\right)^2 - 1\right]\gamma^2 \operatorname{tg}^2\vartheta}\right]^{2k}}{\sqrt{1 - \left[\left(\frac{\widetilde{\beta}}{V}\right)^2 - 1\right]\gamma^2 \operatorname{tg}^2\vartheta}} + \right.$$

$$\left. + \frac{\left[\frac{\widetilde{\beta}}{V} - \sqrt{1 - \left[\left(\frac{\widetilde{\beta}}{V}\right)^2 - 1\right]\gamma^2 \operatorname{tg}^2\vartheta}\right]^2 \left[-\frac{\widetilde{\beta}}{V}\gamma^2 \operatorname{tg}^2\vartheta - \sqrt{1 - \left[\left(\frac{\widetilde{\beta}}{V}\right)^2 - 1\right]\gamma^2 \operatorname{tg}^2\vartheta}\right]^{2n}}{\sqrt{1 - \left[\left(\frac{\widetilde{\beta}}{V}\right)^2 - 1\right]\gamma^2 \operatorname{tg}^2\vartheta}} \right\} \quad (16.14)$$

(when $\frac{\widetilde{\beta}}{V} \geqslant 1$) or

$$N(\vartheta)\sin\vartheta\, d\vartheta = \frac{2k+1}{2} \frac{(\gamma \operatorname{tg}\vartheta)^2}{(1 + \gamma^2 \operatorname{tg}^2\vartheta)^{2k+2}} \times$$
$$\times \left\{ \frac{\left[\frac{\widetilde{\beta}}{V} + \sqrt{1 - \left[\left(\frac{\widetilde{\beta}}{V}\right)^2 - 1\right]\gamma^2 \operatorname{tg}^2\vartheta}\right] \left[-\frac{\widetilde{\beta}}{V}\gamma^2 \operatorname{tg}^2\vartheta + \sqrt{1 - \left[\left(\frac{\widetilde{\beta}}{V}\right)^2 - 1\right]\gamma^2 \operatorname{tg}^2\vartheta}\right]}{\sqrt{1 - \left[\left(\frac{\widetilde{\beta}}{V}\right)^2 - 1\right]\gamma^2 \operatorname{tg}^2\vartheta}} \right\} \quad (16.15)$$

(when $\frac{\widetilde{\beta}}{V} \leqslant 1$).

If $\gamma \gg 1$, $\vartheta \ll 1$, then

$$N(\vartheta)\sin\vartheta\, d\vartheta = 2(2k+1) \frac{\gamma^2 \operatorname{tg}^2\vartheta}{\cos\vartheta} \frac{[\gamma^2 \operatorname{tg}^2\vartheta - 1]^{2k}}{[\gamma^2 \operatorname{tg}^2\vartheta + 1]^{2k+2}}. \quad (16.16)$$

Figure 30 shows the angular distributions $N(\vartheta)$ of particles in the L-system for various k and $\frac{\widetilde{\beta}}{V} = 1.01$. A special feature of this distribution is the presence of two maxima in the case when $k \neq 0$. This circumstance is a consequence of the fact that the angular distribution in the CM-system in this case is similarly characterized by two maxima at $\widetilde{\vartheta} = 0$ and $\widetilde{\vartheta} = \pi$. By transformation from the CM-system to the L-system the directions of propagation of the particles moving in the forward hemisphere in the CM-system are inclined close to the axis, forming a so-called narrow cone, at the same time the particles moving backwards in the CM-system are similarly propagated forwards in the L-system (at sufficiently large energies). However, their velocities are contained within a considerably broader cone. It is appropriate to illustrate a similar picture with simple determinations. Let the angular distribution in the CM-system be symmetrical relative to a plane perpendicular to the axis of motion, but anisotropic in such a way that all the particles lie within the interval of the angles $0, \widetilde{\vartheta}$ and $\pi' - \widetilde{\vartheta}, \pi$. Then, in accordance with (4.3) for $\gamma \gg 1$, the semi-angle at the apex of the narrow

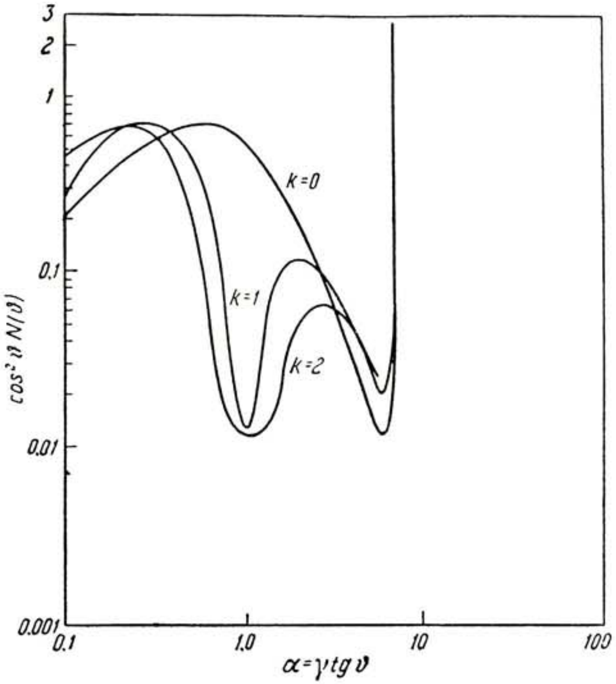


Fig. 30

Angular distribution of particles in the L-system, if their angular distribution in the CM-system is represented by a cosine function (16.13), and their momenta by a δ-function. Parameter $\frac{\tilde{\beta}}{V}$ = 1.01

cone

$$\vartheta_1 = \frac{1}{\gamma} \operatorname{tg} \frac{\tilde{\vartheta}}{2}, \quad (16.17)$$

and the semi-angle at the apex of the broad cone is

$$\vartheta_2 = \frac{1}{\gamma} \operatorname{ctg} \frac{\tilde{\vartheta}}{2}. \quad (16.18)$$

Hence it can be seen that the narrow and broad cones can be

sharply demarcated in the L-system, and this demarcation should be most sharp for small $\tilde{\vartheta}$. The characteristic feature of the distribution considered here is frequently observed using photographic emulsions in investigations of high energy jets, which substantiates the anisotropy of the angular distribution in the CM-system of the secondary particles arising as a result of the collisions.

Section 17. Determination of the Energies of Fast Nucleons

The energy and momentum of the primary particles are the most important characteristics of nuclear reactions. For relatively moderate energies, a number of methods have been developed for determining these values for charged particles (measurement of deflection in a magnetic field, counting the grains in a photoemulsion, measurement of multiple Coulomb). For increase of energy, however, the difficulties of measurement are sharply increased, and for energies of $5 \cdot 10^{10}$ — 10^{11} ev, the usual methods become ineffective. But an appropriate method for the determination of energy has been developed just for such fast particles, based upon a kinematic approach. It consists of measuring the angular distribution of the secondary particles formed as a result of collision of the particles whose energy it is required to determine. [29-31]. The great advantage of this method is the possibility of measuring the energy of a neutral primary particle. In this case it is necessary to determine additionally the direction of their motion, identifying it with the axis of the shower. The determination of the axis of the shower, of course, introduces an additional inaccuracy in the value of the energy (relative to determination of the energy of the charged particles).

Basically, the method depends on relationship (5.5), which we shall write in the form

$$\gamma^2 = \frac{(E_I + m_{II})^2}{(m_I^2 + 2E_I m_{II} + m_{II}^2)}. \quad (17.1)$$

If $E_I \gg m_I$ and $E_I \gg m_{II}$, then

$$\gamma^2 = \frac{E_I}{2m_{II}}. \quad (17.2)$$

On the other hand, the value of γ enters into relationship

(4.1), determining the transformation of the angles of omission from the CM-system into the L-system.

For determination of the value of γ from (17.1) and (4.1), it is necessary, generally speaking, to determine $\tilde{\beta}$ and $\tilde{\vartheta}$ (the value ϑ is determined experimentally), which is scarcely possible. However, in the most important case of very fast particles it can be assumed that $\frac{V}{\tilde{\beta}}=1$, and then relationship (4.4) will no longer depend on the velocity of the particles in the CM-system. It is precisely this substitution which is the essence of the method under consideration.

From (17.2) and (4.4) it follows

$$E_{\rm I}\approx\frac{2m_{\rm II}}{\vartheta^2}\,{\rm tg}^2\,\frac{\tilde{\vartheta}}{2}. \qquad (17.3)$$

Formula (17.3) is essential for determination of energies according to angular distribution. Usually it is applied to a particle producing a shower with a small number (< 3,4) of slow particles (so-called jets). Although there are certain reasons to doubt that these showers are formed as a result of collisions of nucleons with nucleons, however, usually the mass $m_{\rm II}$ is assumed equal to the mass of the nucleon, i.e. $m_{\rm I}=m_{\rm II}=m$. We shall make for the present this assumption* and, furthermore, we shall estimate the possible error associated with it. This it remains to determine the angle $\tilde{\vartheta}$. As a result of the collision of two particles, it is natural to assume that axial symmetry occurs, in the sense that the function $\tilde{\Phi}(\tilde{\vartheta})$ of the angular distribution is independent of the azimuth angle.

As already mentioned, in the collision of two particles it is necessary to assume an even greater degree of symmetry:

$$\tilde{\Phi}(\tilde{\vartheta})=\tilde{\Phi}(\pi-\tilde{\vartheta}).$$

We shall assume first of all that for every particle with a momentus $\tilde{p}$ corresponds a particle with a momentus $-\tilde{p}$. Then it is easy to obtain the relationship which is very often used.

Let ϑ_f and ϑ_{1-f} be the angles within the limits of which are found f and $1-f$ fraction of all particles. Then, from the assumption of symmetry in the CM-system, $\vartheta_{1-f}=\pi-\vartheta_f$

*For the kinematic criteria of nucleon-nucleon collisions see [35].

and consequently, on the basis of relationship (4.4):

$$\gamma^2\approx\frac{1}{{\rm tg}\,\vartheta_f\,{\rm tg}\,\vartheta_{1-f}} \qquad (17.4)$$

or

$$E_{\rm T}\approx\frac{2m}{\vartheta_f\vartheta_{1-f}}. \qquad (17.5)$$

In practice, if $f=\frac{1}{2}$, then

$$E_{\rm T}\approx\frac{2m}{\vartheta_{1/2}^2}.$$

We shall now take into consideration the fact that the conditions for the existence of two particles moving through complementary angles ($\tilde{\vartheta}$ and $\pi-\tilde{\vartheta}$) are achieved only on the average. In this case it is necessary to assign some particular form to the function Φ and to the function for correlation of the angles of emission of the particles. We shall find the maximum probability of the value of the energy for the given function Φ, assuming that the angles of emission of the particles are statistically independent [30, 31]. At the present time there are no experimental data which would verify the accuracy of this assumption. We shall see it as being the most simple assumption, bearing in mind the necessity for verification of the results by experiment*.

In this case, the probability Φ of the particles being emitted at the angles $\vartheta_1\ldots\vartheta_N$, i.e. equal to [30]:

$$\Phi(\vartheta_1\ldots\vartheta_N)=\prod_{i=1}^{N}\Phi(\vartheta_i)\,d\vartheta_i, \qquad (17.6)$$

where ϑ_i, $d\vartheta_i$ are related to $\tilde{\vartheta}_i$, $d\tilde{\vartheta}_i$ by the following relationships (see (4.4))

$$\cos\tilde{\vartheta}=\frac{1-\gamma^2\,{\rm tg}^2\,\vartheta}{1+\gamma^2\,{\rm tg}^2\,\vartheta}, \qquad (17.7)$$

$$d\tilde{\vartheta}=\frac{2\gamma\sec^2\vartheta}{1+\gamma^2\,{\rm tg}^2\,\vartheta}\,d\vartheta. \qquad (17.8)$$

Hence, the maximum probable value of the quantity $\gamma=\gamma_{\rm B}$

*This supposition is adequate, of course, if the number of secondary particles is large and if there are no decay products of secondary particles among them.

(consequently also the energy) is determined by the equation

$$\left\{\frac{\partial}{\partial\gamma}\Phi(\vartheta_1\ldots\vartheta_N)\right\}_{\vartheta_i=\mathrm{const}}=\sum_{i=1}^{N}\left\{\frac{\partial}{\partial\gamma}\ln\Phi(\vartheta_i)\right\}_{\vartheta_i=\mathrm{const}}. \quad (17.9)$$

Dilworth et al. [30] have calculated γ_B, assuming that

$$\Phi(\tilde{\vartheta}_i)=a|\cos\tilde{\vartheta}|^k\sin\tilde{\vartheta} \qquad (k=\mathrm{const}).$$

In this case γ_B can be found from the equation

$$\frac{N}{2\gamma_B^2}=\sum_{i=1}^{N}\left[\frac{1}{\cos^2\vartheta_i+\gamma_B^2}-\frac{k}{2}\left\{\frac{1}{1+\gamma_B^2\mathrm{tg}^2\vartheta_i}-\frac{1}{1-\gamma^2\mathrm{tg}^2\vartheta_i}\right\}\right]. \quad (17.10)$$

We shall also find the spread of the value γ.

Let us write (4.4) in the form

$$\ln\gamma=-\ln\mathrm{tg}\,\vartheta+\ln\frac{\mathrm{tg}\,\tilde{\vartheta}}{2}. \quad (17.11)$$

Summing over the angles of all secondary charged particles we obtain

$$\ln\gamma=-\frac{1}{N}\sum_{i=1}^{N}\ln\mathrm{tg}\,\vartheta_i+\frac{1}{N}\sum_{i=1}^{N}\ln\mathrm{tg}\frac{\tilde{\vartheta}}{2} \quad (17.12)$$

(In the given case N is the number of secondary charged particles in the shower.)

For the maximum simplification we shall use the fundamental limiting theory of probability, according to which the sum N of several random values for $N\to\infty$ is also a random value, the spread of which is represented by a Gaussian function with a dispersion of $\sigma^2 N$, where σ^2 is the mean square deviation of the primary random value*. Therefore (17.12) can be represented in the form

$$\ln\gamma=-\frac{1}{N}\sum_{i=1}^{N}\ln\mathrm{tg}\,\vartheta_i\pm\frac{\sigma}{\sqrt{N}}, \quad (17.13)$$

where

$$\sigma^2=\int\left(\ln\mathrm{tg}\frac{\tilde{\vartheta}}{2}\right)^2 J\left(\ln\mathrm{tg}\frac{\tilde{\vartheta}}{2}\right)d\left(\ln\mathrm{tg}\frac{\tilde{\vartheta}}{2}\right). \quad (17.14)$$

and the function J is the standard deviation of the value $\ln\mathrm{tg}\frac{\tilde{\vartheta}}{2}$ in the CM-system.

Naturally the value of σ depends on the degree of anisotropy of the angular distribution in the CM-system. However, for approximate calculations the value of σ can be taken as equal to 1.

*This supposition is adequate, of course, if the number of secondary particles is large and if there are no decoy products of secondary particles among them.

As mentioned, the methods described here for determining energies are adequate for the case when the ratio $\frac{\tilde{\beta}}{V}$ approaches 1. In practice this implies that the energy of the primary particle is greater, and consequently as a rule, also the number of secondary particles is greater than 1. It is impossible to determine the precise limit of the method, however it can be relied upon in every case to give an accurate order of magnitude for the energy of the primary particle for energies of $E_n>10^{11}$ ev.

An estimation of the error, due to the fact that the velocities of the secondary particles do not satisfy the above-mentioned conditions, can be carried out in several ways, however, all things equal, there remains the difficulty associated with a small number of secondary particles. We shall note here a few of these methods. Thus, it is possible to use the method of successive approximations, having assumed in the first approximation $\frac{\tilde{\beta}}{V}=1$, and to determine in this manner a certain γ_0, then according to formula (4.7) to evaluate $\tilde{\beta}$ and to carry out all the calculations again. Naturally, as a result of this, it is necessary to determine independently the momentum and mass of certain secondary particles. Another method of determination of energy for the case when the momenta and masses of certain secondary particles are known is symmetrization of the angular distribution [22]. This method boils down to a determination in accordance with formula (17.1) of a value of γ, for which, in the corresponding system of co-ordinates, the angular distribution has maximum symmetry relative to the plane perpendicular to the direction of motion.

An additional substantial source of error can be the energy distribution of the primary particles. In this case (as happens in cosmic rays), when the energy spectrum of the particles causing the showers is represented by a

*For the derivation of the theorem, it is assumed that the dispersion σ^2 exists. Moreover, generally speaking the Gaussian function so obtained is displaced relative to zero; this function is symmetrical relative to zero in one important actual case, when function $\Phi(\vartheta)$ is symmetrical, i.e. it is fulfilled by the condition $\Phi(\vartheta)=\Phi(\pi-\vartheta)$. This also is the second condition for applicability.

rapidly falling function, even insignificant fluctuations in showers of definite energy can give rise to a large error. The reason is that the primary particles of low energies are more frequent in the spectrum and consequently, are registered with a greater probability than particles with higher energies. Therefore, the method in the form described above gives good results only in the case when the spectrum of the primary particles is constant or almost constant. For the study of cosmic rays it is necessary, in addition to the effect of the spectrum, to take into account the relationship between the total number of particles in a star and the magnitude of the energy. This problem is analysed in detail in the work [32].

PART TWO
QUANTUM THEORY

In this part of the book we shall consider the consequences of the general properties of space-time for nuclear reactions, by means of quantum mechanics. These consequences, as we shall see below, prove to be considerably more important than in classical mechanics. It is essential to emphasize at the very beginning that our problem is to discriminate from amongst all the properties of a reaction those which are consequences of the established laws of nature. Such a discrimination proves to be very useful. It permits the determination of the requisite number of actual parameters of a reaction (generalized phase shift analysis), and the association of the various processes with rigid relationships. Moreover, it assists the assessment of experimental data and permits the most important characteristics of particles to be established (their spin, parity, isotopic spin).

In the main we shall use non-relativistic quantum mechanics. Our treatment assumes a knowledge by the student of quantum mechanics to the extent of a University course or D.I. Blokhintsev's book "Foundations of Quantum Mechanics", to which we shall refer frequently, and does not demand a knowledge of Group Theory. In certain places we shall indeed be obliged to use results from Group Theory, but this does complicate the understanding of the physical significance of the theory set forth below or the application of the general formulae to analysis of experimental data. In connexion with this, a small number of results will be presented without proof. For students interested in this side of the work, we shall refer to the appropriate section of the book by L. Landau and E. Lifschitz "Quantum Mechanics" or to a special review of articles from periodical literature.

We shall construct our account on the basis of the fundamental concept of the S-matrix* which is of great significance in the theory of elementary particles and in nuclear physics. Therefore the ability to work easily with this concept becomes a necessity to every qualified experimentor for whom the present book is designed. In this part of the book we shall apply the formulation of quantum mechanics advanced by Dirac**. This terminology and notation is a most appropriate form of quantum mechanics, applied broadly in theoretical works, which illustrates in concise form the significance of the various coefficients encountered in the theory of angular distributions, correlations and other problems, and so facilitates working with these coefficients.

*This method of discussion of the formal theory of nuclear reactions is used in [23 and 24].

**It is assumed that the student has a knowledge of the theory of the introduction to quantum mechanics to the extent of Chapter VII of [1].

CHAPTER VI

THE SCATTERING MATRIX AND ITS PROPERTIES

Section 18. The S-matrix

The general formulation of the problem in the case of a nuclear reaction consists in the comparison of the properties of the particles and of the parameters by which the state of their motion is described prior to the reaction, with the parameters and properties of the reaction products. In the case when the interaction and motion of the particles can be described by means of classical mechanics, one speaks of comparison of the initial and final co-ordinates, of the momenta of the particles, and of any variables characterizing their internal state. In quantum mechanics we should speak about the comparison of the initial and final state of the system. But the state, as is well-known, is given by a set of quantum numbers. Thus, it is necessary for us to find a rule, which allows us to compare the quantum numbers describing the initial with those of the final state of the system. For example, in the case of scattering of spinless particles by a powerful field, we can speak of a rule connecting the quantum numbers which characterize the orbital angular momentum l, its projection m in a certain direction, and the energy E in the initial and final states. We observe that the state of motion of a spinless particle can also be given by its momentum $\boldsymbol{p}$.

In the case of interaction of two spinless particles, the following sets of quantum numbers can be given $(l_1, l_2, m_1, m_2, E_1, E_2)$ or $(\boldsymbol{p}_1, \boldsymbol{p}_2)$. If the particles have spin, then in the corresponding set it is necessary to include the numbers s and μ, characterizing the magnitude of the spin and its projection. If, as a result of collision, other particles are created or the internal state of the colliding particles is changed (such a collision is customarily called inelastic), then in the number of variables it is necessary to include further quantum numbers characterizing the structure of the particles, their internal state.

Since the study of the structure of the particles does not enter into our task, we shall denote all these other characteristics of our system by the single index α.

In accordance with quantum mechanics, the state of a system is described by the wave function $\Psi_n(x)$, where n is the index of state, i.e. the short symbol for the set of quantum numbers proposed above, giving the state of the system, and x is the index of representation, i.e. a set of variables upon which the wave function depends. The wave function satisfies Schrödinger's Equation*.

$$i\frac{\partial\Psi}{\partial t}=\hat{H}\Psi. \qquad (18.1)$$

Equation (18.1), defining the increment Ψ in a certain small interval of time Δt can be written down in the following manner:

$$\Psi(t+\Delta t)=\Psi(t)-i\int_{t}^{t+\Delta t}\hat{H}\Psi(t)\,dt. \qquad (18.2)$$

Having established this equation, we shall introduce the operator $\hat{U}(t+\Delta t, t)=1-i\hat{H}\Delta t$, which transforms the value of the wave function from that at instant of time t to that at the instant of time $t+\Delta t$. An integral operator $\hat{U}(t, t_0)$ can also be introduced, converting $\Psi(t_0)$ into $\Psi(t)$, where $t-t_0$ is the final value. It is easy to prove that**

$$i\frac{\partial\hat{U}}{\partial t}=\hat{H}\hat{U}. \qquad (18.3)$$

Let us consider a reaction of the type

$$\mathrm{I}+\mathrm{II}\rightarrow 1+2, \qquad (18.4)$$

i.e. as a result of collision of particle I with particle II, particles 1 and 2 are formed, having, generally speaking, an altogether different characteristic and even another nature.

*The cap over the letter denotes that the corresponding quantities are operators. Here and in future $\hbar=c=1$.
**See for example [3].

Equation (18.1) describes the entire process of interest to us, i.e. there exists such a solution Ψ, which for $t\rightarrow-\infty$ transforms into the wave function $\Psi_i(x)$, describing the two non-interacting particles I and II by specific characteristics of motion and the internal state - i, and for $t\rightarrow+\infty$ transforms into the wave function Ψ_f for the two non-interacting particles 1 and 2.

Since in practice the states of non-interacting particles are recorded prior to and after collision, i.e. for $t=-\infty$ and $t=+\infty$, then the operator transforming $\Psi(-\infty)$ into $\Psi(+\infty)$ will be of interest to us

$$\lim_{\substack{t_0\to-\infty\\ t\to+\infty}}\hat{U}(t, t_0)=\hat{S},$$
$$\Psi_f(+\infty)=\hat{S}\Psi_i(-\infty). \qquad (18.5)$$

The operator $\hat{S}$ is called the S-matrix.

For calculating $\hat{U}$ or $\hat{S}$, as is evident from our discussions, it is necessary, in general, not only to know $\hat{H}$ (usually the operator $\hat{H}$ is of course unknown for interaction of nuclear particles), but also to be able to solve the Schrodinger equation for this value of $\hat{H}$. At first sight we haven't in the least simplified the problem. The advantage of introducing the operator $\hat{S}$ consists in that the matrix elements of $\hat{S}$ are very simply related to the observed characteristics of the process - the probability of transition, and that the general laws of nature are directly expressed in the properties of the operator $\hat{S}$. That is, by means of the S-matrix we can solve the problem formulated in the introduction - to discriminate from amongst all the properties of the reaction those properties which are consequences of the established laws of nature.

We shall derive the general properties of the S-matrix.

We shall show that the squares of the matrix elements of the S-matrix determine the probability of finding one or other value of the dynamic transitions in the final state, for a specified initial state.

We shall resolve $\Psi_i(-\infty)$ and $\Psi_f(+\infty)$ into eigen values of the operator $\hat{Q}$, describing any dynamic transition (angular momentum, force etc.):

$\hat{Q}\Psi_q = q\Psi_q$, q is the eigen value of the quantity $\hat{Q}$.

$$\Psi_i(-\infty) = \sum_q C_q^i \Psi_q; \qquad \Psi_f(+\infty) = \sum_q C_q^f \Psi_q.$$

Substituting these in equations (18.5) we obtain

$$C_{q'}^f = \sum_q S_{q'q} C_q^i.$$

If, in the initial state the quantity q had a definite value q_0, i.e. $C_q^i = \delta_{qq_0}$ *, then

$$C_{q'}^f = S_{q'q_0}.$$

But, according to the general ideas of quantum mechanics, $|C_{q'}^f|^2$ is the probability of finding the value $q = q'$ in the state f; therefore $|S_{q'q}|^2$ determines the probability of transition from the state q_0 into the state q'.

Section 19. Unitarity of the S-matrix

We recall that a matrix is called unitary, if it possesses the properties

$$\sum_n S_{n\alpha}^* S_{n\beta} = I_{\alpha\beta}; \quad \sum_\alpha S_{m\alpha} S_{n\alpha}^* = I_{mn},$$

or more briefly, in symbolic inscription

$$\hat{S}^+ \hat{S} = \hat{I} \text{ and } \hat{S}\hat{S}^+ = \hat{I}; \qquad (19.1)$$

(the asterisk stands for a complex conjugate, the plus sign for a Hermitian conjuate); $\hat{I}$ is here and henceforth a unit diagonal matrix. This property arises from the orthogonality and normalization of the wave functions:

$$(\Psi_m, \Psi_n) = I_{mn}. \qquad (19.2)$$

* δ_{qq} as usual denotes the Kronecker delta:

$$\delta_{qq_0} = \begin{cases} 1 \text{ for } q = q_0 \\ 0 \text{ for } q \neq q_0 \end{cases}.$$

For simplicity here we shall consider the value with a discrete spectrum. The discussion of the case of a continuous spectrum of the value q is obvious by generalization.

Under the scalar product we shall assume (as usual) the summation and integration with respect to every index x (the index of representation) upon which the wave function depends; I_{mn} is an abbreviation which describes a δ-function if the quantum numbers have a continuous spectrum and a Kronecker delta if the quantum numbers have a discrete spectrum. The relationship (19.2) leads to*:

$$\frac{\partial}{\partial t}(\Phi, \Psi) = \left(\frac{\partial \Phi}{\partial t}, \Psi\right) + \left(\Phi, \frac{\partial \Psi}{\partial t}\right) = i(\Phi, \hat{H}\Psi) - i(\Phi, \hat{H}\Psi) = 0.$$

Let us consider the consequences arising from (19.2) for the operator $\hat{U}(t, t_0)$ introduced above:

$$\Psi_\alpha(t) = \sum_n U_{\alpha n}(t, t_0) \Psi_n(t_0),$$

$$\Psi_\beta^*(t) = \sum_m U_{\beta m}^*(t, t_0) \Psi_m^*(t_0),$$

$$(\Psi_\beta, \Psi_\alpha) = \delta_{\alpha\beta} = \sum_n U_{\beta n}^* U_{\alpha n} = \sum_n U_{n\beta}^+ U_{\alpha n} = (\hat{U}\hat{U}^+)_{\alpha\beta}.$$

For $t_0 \to -\infty$ and $t \to +\infty$, we obtain the very important result

$$\hat{S}\hat{S}^+ = \hat{I}.$$

This equation has a simple physical significance. In fact, taking the diagonal elements from both its members we obtain

$$\sum_n |S_{\alpha n}|^2 = 1. \qquad (19.3)$$

Since $|S_{\alpha n}|^2$ defines the probability of transition from the state n into state α, then equation (19.3) simply denotes that the sum of the probabilities of all transitions is equal to 1. Hence it is obvious, that by omitting any state in the sum of (19.3), we shall underestimate its value Equations (19.1) impose a strict limit on the S-matrix for different processes. From these limits there arise a whole series of relationships between the cross-sections of different processes, which we shall consider below.

*We point out that $\hat{H}$ is Hermitian.

Section 20. Constants of the Motion

By constants of motion in quantum mechanics are implied dynamic variables, the operators of which $\hat{Q}$ satisfy the equation:

$$\frac{d\hat{Q}}{dt}=\frac{\partial\hat{Q}}{\partial t}+[\hat{H}\hat{Q}]=0.$$

If $\dot{Q}$ evidently does not depend on time, then this equation is reduced to a state of the commutativity of $\hat{Q}$ with the Hamiltonian. It is easy to see that such operators $\hat{Q}$ also commute with the operator $\hat{S}$. Actually, $\hat{S}$ by our definition, comprises an infinitely large number of infinitely small operators, each one of which is proportional to $\hat{H}$, i.e.

$$[\hat{S}\hat{Q}]=0. \qquad (20.1)$$

But commutating operators can be reduced to a diagonal form* Consequently, if we choose constants of motion (angular momentum, momentum, energy, isotopic spin etc.) as the indices α and n, then the S-matrix will be diagonal with respect to this index:

$$(q'\gamma'|S|\gamma q)=(\gamma'|S^q|\gamma)\,\delta_{q'q}\ ^{**}). \qquad (20.2)$$

where γ includes all the remaining quantum numbers with the exception of q. This relationship is the basis of the theory of correlation in angular distributions and other important phenomena. The index q denotes that $(\gamma'|S^q|\gamma)$, depends upon q.

It should be emphasized that not all the constants of motion commute amongst themselves (e.g. angular momentum and momentum, different projections of angular momentum). This means, on the one hand, that it is impossible to select simultaneously as indices n, all constants of motion. On the other hand, this means that $\hat{S}$ plainly does not depend upon certain quantum numbers. We shall show, for example, that the S-matrix does not depend upon the quantum number characterizing the projection of the total angular momentum. $\hat{S}$ commutes with all the operators of the projection of total angular momentum

$$[\hat{J}_x\hat{S}]=0,\quad [\hat{J}_y\hat{S}]=0,\quad [\hat{J}_z\hat{S}]=0. \qquad (20.3)$$

*See [1], page 94.

**We shall write down the matrix elements in accordance with Dirac's notation: $S_{q'q}\equiv(q'|S|q)$.

We shall denote the quantum number for the projection of the total angular momentum on the z axis by M, and write it in matrix form using equation (20.3). Taking into account that $\hat{J}_x\Psi_M\neq\mathrm{const}\,\Psi_M$, we obtain

$$\sum_{M''}(M'|J_x|M'')(M''\gamma'|S|\gamma M)=\sum_{M''}(M'\gamma'|S|\gamma M'')(M''|J_x|M).$$

But according to equation (20.2), the S-matrix diagonal with respect to M:

$$(M'|J_x|M)(\gamma'|S^M|\gamma)=(\gamma'|S^{M'}|\gamma)(M'|J_x|M),\quad M'\neq M.$$

Hence it follows that $(\gamma'|S^M|\gamma)$ is independent of M. This result emerges from the property of isotropy of space (a consequence of which is the Law of Conservation of angular momentum). Actually, as was shown above, $|S_{\alpha n}|^2$ characterizes the probability of transition and is also independent of the choice of the reference system, and the quantum number M (projection of angular momentum) is changed by a simple reversal of the co-ordinate system.

In a similar manner $(\gamma'|S^q|\gamma)$ is independent of the total momentum of the system, of the projection of the total isotopic spin of the system and so forth. Briefly, $\hat{S}$ is independent of those quantum numbers which are changed as a result of transformations under which the Hamiltonian of the system is invariant. The profound relationship should be emphasized between transformations leaving the Hamiltonian of the system invariant and the constants of motion. A detailed discussion of this relationship is beyond the bounds of the present book*.

Relationship (20.2) is used below for discussion of the following constants of motion: energy, momentum, angular momentum, its projection, parity. It is assumed that the basic properties of these constants of motion are well known to students**. In addition, as an example we shall consider

*For the relationship between transformations of a system of co-ordinates leaving the Hamiltonian of the system invariant, and such constants of motion as angular momentum and momentum, see [2], Sections 13 and 24.

**See [1], Sections 25, 103 and [2], Sections 24-28.

one further important constant of motion - isotopic spin. The student can become acquainted with this concept through [9]. It should be noted that adaptations of (20.2) are extensively used in the literature, and also applied to other constants of motion (intrinsic parity, strangness etc.), the consideration of which lies somewhat aside from our theme. We shall not discuss the weak interaction of particles liable to disintegration, the study of which led to the discovery of the non-conservation of parity.

Section 21. Time Reversal*

If the system is not within an external field, then all instants of time for such a system are of equal right as are also all directions of space. In classical and quantum mechanics this circumstance leads to the Law of Conservation of Energy. In addition, in classical mechanics equations of motion are invariant relative to the substitution $t \to -t$. Let us, for example, consider a solution of a Newtonian equation describing the motion of a system of material points. Suppose that at the instant of time $t = t_1$ the radii-vector of the points and their velocities are equal to $r_i(t_1)$, $v_i(t_1)$ and after the lapse of an interval of time $\Delta t = t_2 - t_1$, at the instant t_2 these dimensions have the values $r_i(t_2)$, $v_i(t_2)$. The invariancy of the equation relative to the substitution $t \to -t$ means that a solution exists characterizing the fact that the radii-vectors and velocities of the material points $r_i(t_2)$, $-v_i(t_2)$, are transformed in the same arbitrarily chosen interval of time into $r_i(t_1)$, $-v_i(t_1)$. Not all systems possess such symmetry. Consider a system of charged particles in a magnetic field. In this case as is well-known (see for example [11]), it is necessary to include, in the operation of time conversion, the reversal of the direction of the magnetic field. If this is not done, then the reversibility of time does not hold for the system. Since classical mechanics is the limiting case of quantum mechanics, it should be expected that time reversibility finds its counterpart in quantum mechanics. Let us consider the quantum mechanics of

*This paragraph is designed only for the more advanced student. For this study one should be familiar with Sections 22 and 27. It can be omitted for a first reading without loss of understanding.

conservation, i.e. the Hamiltonian is independent of time. The Schrödinger equation

$$i \frac{\partial \Psi(t)}{\partial t} = \hat{H}\Psi(t) \quad (21.1)$$

for a time $t \to -t$ transforms into

$$-i \frac{\partial \Psi(-t)}{\partial t} = \hat{H}\Psi(-t) \quad (21.2)$$

(21.2) does not coincide with (21.1). This does not permit us to designate $\Psi(-t)$ as a time reversal solution of equation (21.1). For calculation of the latter we consider the complex conjugate of (21.2):

$$i \frac{\partial \Psi^*(-t)}{\partial t} = \hat{H}^*\Psi^*(-t). \quad (21.3)$$

Since $\hat{H}$ is Hermitian it follows that $\hat{H}$ and $\hat{H}^*$ have the same proper value (but generally speaking, different proper functions); this means that a unitary operation $\hat{V}$ exists such that

$$\hat{V}\hat{H}^*\hat{V}^+ = \hat{H}. \quad (21.4)$$

From (21.3) by means of relationship (21.4) we obtain

$$i \frac{\partial [\hat{V}\Psi^*(-t)]}{\partial t} = \hat{H}[\hat{V}\Psi^*(-t)]. \quad (21.5)$$

Equating (21.5) with (21.1), we see that $\hat{V}\Psi^*(-t)$ naturally qualifies as a solution by time reversal. Thus, time reversal transformation contains a complex conjugate

$$\hat{T}\Psi(t) = \hat{V}\hat{K}\Psi(-t); \quad (21.6)$$

Here, $\hat{T}$ is the operator for the time reversal, and $\hat{K}$ is the operator for the complex conjugate.

We shall consider the Law of the time reversal transformation of operators and we shall assume that the wave functions for such a transformation maintain their form, and that the operators of the dynamic quantities are changed (c.f. we are changing from Schrödinger's representation to Heisenberg's representation [1]).

For such a transformation it follows from the general formula of quantum mechanics

$$(\hat{T}\Psi(t),\ \hat{Q}\hat{T}\Psi(t)) = (\Psi(-t),\ \tilde{\hat{Q}}\Psi(-t)),$$

where $\tilde{\hat{Q}}$ is the transformation operator (time reversal). Thus, for operators independent of time we have

$$\hat{V}^{+}\hat{Q}\hat{V} = \tilde{\hat{Q}}^{\tau}. \qquad (21.7)$$

The symbol "T" signifies that the corresponding operator if transposed. If $\tilde{\hat{Q}} = \hat{Q}$, as happens for $\hat{H}$ in accordance with (21.4) and the Hermitian property of $\hat{H}$, then we say that this operator is invariant relative to time reversal $(t \to -t)$.

It is not difficult to show that from the invariancy of $\hat{H}$ relative to the operation $t \to -t$, the invariancy of the S-matrix also follows:

$$\hat{V}^{+}\hat{S}\hat{V} = \hat{S}^{\tau}. \qquad (21.8)$$

Relationship (21.8) is terms the theorem of reciprocity. It indicates a very general property of the S-matrix. However, (21.8) itself represents a small supporting confirmation whilst a clear idea of the operator $\hat{V}$ is unknown. In order to find $\hat{V}$ it is necessary to assign transformation properties for the operators of the dynamic variables constituting the entire set of values characterizing the system. This determines the transformation properties of every other dynamic variable. The sources of our knowledge concerning the operators of quantum mechanics are the principles of co-ordination and experiment. Consequently, we shall superimpose on $\hat{V}$ a requirement that the values not changing sign as a result of the transformation $t \to -t$ in classical mechanics (co-ordinates, energy etc.), should have operators which are invariant relative to the transformation (21.7). Indeed, the values changing sign as a result of the transformation $t \to -t$ in classical mechanics (velocity, momentum, angular momentum, vector potential, electro-magnetic field etc.) should have operators obeying the condition

$$\hat{V}^{+}\hat{Q}\hat{V} = -\hat{Q}^{\tau}. \qquad (21.9)$$

The operators of spin, being dimensions analagous to the angular momentum should also be transformed in accordance with (21.9). If the operator $\hat{V}$ exists, possessing these properties, then by definition the motion of the system is reversible in time.

Let us find an explicit form for the matrix operator $\hat{V}$ for a state, of specified quantum numbers for angular momentum $l,\ l'$ (for example, orbital angular momentum of one of the particles of the system) and its projections m and m':

$$(l'm'\alpha' \,|\, V \,|\, lm\alpha),$$

here α and α' are the remaining quantum numbers.

From the conditions imposed upon $\hat{V}$ for the requirement of time reversal and that the operators under discussion are Hermitian,

$$\hat{V}^{+}\hat{l}^{2}\hat{V} = \hat{l}^{2}, \qquad (21.10)$$

$$\hat{V}^{*}\hat{l}\hat{V} = -\hat{l}^{*}. \qquad (21.11)$$

From (21.10) it follows that $\hat{V}$ is diagonal with respect to the quantum number l. The dependency of $\hat{V}$ on the quantum numbers m and m' can be established by solving equation (21.11) using an explicit expression for the matrices of the projections of the angular momentum (see [2], Section 25):

$$(l'm'\alpha' \,|\, V \,|\, lm\alpha) = (\alpha' \,|\, V^{l} \,|\, \alpha)\, \delta_{l'l}\delta_{-m'm} e^{i\pi m}. \qquad (21.12)$$

Using (21.7) for all the remaining dynamic variables (squares of angular momenta, spins, momenta, etc.), (21.12) can be written in the form:

$$(l'm'\alpha' \,|\, V \,|\, lm\alpha) = \delta_{\alpha'\alpha}\delta_{l'l}\delta_{-m'm} e^{i\pi[\beta(\alpha,\, l)+m]},$$

where the phase β is arbitrary; it can be chosen so as to simplify the calculations. Generally, β is chosen such that

$$(l'm'\alpha' \,|\, V \,|\, lm\alpha) = \delta_{\alpha'\alpha}\delta_{l'l}\delta_{-m'm} e^{i\pi(l+m)}. \qquad (21.13)$$

The first reason for our choice of β is the fact that the matrix $\hat{V}$ is real; the second reason follows from a consideration of the addition of angular momenta (see Section 27):

$$\Psi_{AJM} = \sum_{\mu_1\mu_2} (JM \,|\, j_1\mu_1 j_2\mu_2)\, \Psi_{\alpha_1 j_1\mu_1}\Psi_{\alpha_2 j_2\mu_2},$$

Ψ_{AJM} is the wave function of the total angular momentum. Into the index A of this function enters $j_1, \alpha_1, j_2, \alpha_2$. If each one of the states $\Psi_{\alpha_1 j_1 \mu_1}$ and $\Psi_{\alpha_2 j_2 \mu_2}$, are transformed as a result of substituting $t \to -t$ by means of (21.13), then using the property of vector addition coefficients (see Section 27):

$$(J-M \mid j_1 - \mu_1 j_2 - \mu_2) = (-1)^{j_1+j_2-J}(JM \mid j_1\mu_1 j_2\mu_2),$$

it can be shown that Ψ_{AJM} is also transformed in accordance with (21.13).

Thus, as a result of our choice of β, there is no necessity to bother about transformation of the phase - all the wave functions of angular momenta and spins are transformed in a like manner.

Let us consider the S-matrix in the representation which is determined by the total angular momentum J, its projection M and the dynamic variables, invariant relative to the transformation $t \to -t$. For example, for a reaction of the type $a+X \to b+Y$, in accordance with the preceding paragraph we shall have

$$(s'l'\alpha' \mid S^J \mid sl\alpha)\,\delta_{JJ'}\delta_{MM'},$$

here s, s' are the total spins of the particles, l', l are the orbital angular momenta of the relative motion. Substituting this expression in (21.8) and using the explicit form of the matrix operator $\hat{V}$ (21.13), we find that the S-matrix in such a presentation is symmetrical:

$$(B \mid S^J \mid A) = (A \mid S^J \mid B). \qquad (21.14)$$

This result reflects one of the fundamental properties of the S-matrix, having considerable value in applications.

Similarly the equally important relationship,

$$\begin{aligned}(p_f j_1\mu_1 j_2\mu_2\alpha' \mid S \mid p_i j_I\mu_I j_{II}\mu_{II}\alpha) = \\ = (-1)^{j_I+\mu_I+j_{II}+\mu_{II}+j_1+\mu_1+j_2+\mu_2} \times \\ \times(-p_i j_I - \mu_I j_{II} - \mu_{II}\alpha \mid S \mid -p_f j_1 - \mu_1 j_2 - \mu_2\alpha'),\end{aligned} \qquad (21.15)$$

can be proved, here j_i, μ_i are the spins of the particles and their projections; p_i and p_f are the momenta of the relative motion in the initial and final states.

If the function Ψ is transformed by a unitary matrix $\hat{W}$:

$$\Psi' = \hat{W}\Psi, \quad (\hat{T}\Psi)' = \hat{W}\hat{T}\Psi,$$

then from (21.6) and equation

$$(\hat{T}\Psi)' = \hat{V}'\Psi'^*$$

we find

$$\hat{V}' = \hat{W}\hat{V}\hat{W}^{\tau}, \qquad (21.16)$$

i.e. $\hat{V}$ is transformed from one representation into another (see Section 22) not as the general unitary matrix, but by means of a transposing transformation.

Since the form of $\hat{V}'$ can be found from consideration of invariancy, the choice of phase for $\hat{V}$ implies an unambiguous choice of phase for the transforming matrix $\hat{W}$.

Let us consider an example. Let $\hat{V}$ be fixed by (21.13) and let $\hat{W}$ transform it from the representation lm into a representation assigned the single vector $\boldsymbol{n}$:

$$W = e^{i\gamma_l} Y_{lm}(\boldsymbol{n}), \qquad (21.17)$$

where γ_l is the phase factor.

The vector $\boldsymbol{n}$ can be directed with respect to $\boldsymbol{r}$ (the radius vector of the particle) or with respect to $\boldsymbol{p}$ the momentum of the particle). From the requirement of time reversal ($\hat{V}^+\hat{p}\hat{V} = -\hat{p}^*$; $\hat{V}^+\hat{r}\hat{V} = \hat{r}$) we have

$$(\boldsymbol{p}' \mid V \mid \boldsymbol{p}) = \delta(\boldsymbol{p}'+\boldsymbol{p}); \quad (\boldsymbol{r}' \mid V \mid \boldsymbol{r}) = \delta(\boldsymbol{r}'-\boldsymbol{r}). \qquad (21.18)$$

We shall carry out transformation (21.17) on $\hat{V}$, having the form (21.13), in accordance with formula (21.16):

$$\begin{aligned}(\boldsymbol{n}' \mid V \mid \boldsymbol{n}) &= \sum_{lml'm'} e^{i\gamma_{l'}} Y_{l'm'}(\boldsymbol{n}')\,\delta_{l'l}\delta_{-m'm} e^{i\pi(l+m)} e^{i\gamma_l} Y_{lm}(\boldsymbol{n}) = \\ &= \sum_{lm} e^{2i\gamma_l + i\pi l} Y^*_{lm}(\boldsymbol{n}') Y_{lm}(\boldsymbol{n}) = \begin{cases} \delta(\boldsymbol{n}'+\boldsymbol{n}) & \text{for } \gamma_l = 0 \\ \delta(\boldsymbol{n}'-\boldsymbol{n}) & \text{for } \gamma_l = -\frac{\pi l}{2}. \end{cases}\end{aligned}$$

Comparing this with (21.18) we see that our choice of phase in (21.13) corresponds to the wave functions of the orbital momentum $Y_{lm}\left(\frac{p}{p}\right)$, in the p-representation and $e^{-i\frac{\pi l}{2}} \times$ – $Y_{lm}\left(\frac{r}{r}\right) = i^{-l} Y_{lm}\left(\frac{r}{r}\right)$ in the r-representation.

Section 22. Transformation Functions

We have examined the most important properties of the S-matrix. Now it is necessary for us to establish in what manner these properties are connected with experimentally observed values, for example with cross-sections, angular distributions and so forth.

It was mentioned above that the squares of the matrix elements of the S-matrix determine the probability of transition from a particular state of the initial set of quantum numbers i, into a particular state of the final set f. This means, that if we choose as quantum numbers the angles θ, φ, determining the direction of flight of the particle, then the square of the S-matrix will give the probability density of detecting the particles in a given direction. Further, if we choose the quantum numbers l, m, then we obtain the probability of detecting particles with a specific value and projection of the angular momentum.

We shall now deal with the transformation of the S-matrix defined by a given set of quantum numbers, into the S-matrix defined by another set. Such transformations will be extremely useful to us in future. We shall cite an example. The Laws of Conservation impose a number of limitations on the form of the S-matrices. If, in the set of the quantum numbers α and n there occur the quantum numbers of constants of motion, then the S-matrix is diagonal. If the limitations imposed by the laws of conservation are of interest to us, for example in angular distributions, then we should transform the S-matrix from the representation specified by the quantum numbers of the constants of motion, into the representation specified by the angles.

In accordance with the general principles of quantum mechanics [1-3], in order to find the probability of the variable q, having a particular value, it is necessary to resolve $\Psi_n(x)$ into eigen-function of the corresponding operator $\hat{Q}$,

$$\Psi_n(x) = \sum_q C_q^n \Psi_q(x) \tag{22.1}$$

and to take the square of the modulus of the coefficient of resolution C_q^n. The set of coefficients C_q^n is a wave function in the "q-representation".

Expression (22.1) can be written down in Dirac's notation thus:

$$(x \mid n) = \sum_q (x \mid q)(q \mid n). \tag{22.2}$$

Hence it follows that the eigen-function of the operator $\hat{Q}$ $\Psi_q(x) \equiv (x \mid q)$ is the transformation function from the q-representation into the x-representation.

The convenience of Dirac's notation consists in the fact that they reflect, with the maximum accuracy and in the most general form, the fundamental principles of quantum mechanics. In fact, the notation of the transformation function $(x \mid q)$ stresses a certain symmetry between the indices of the x-representation and the indices of the q-state. Moreover, this system of notation permits us to explain in a simple form the significance of the various coefficients encountered in the theory of angular distributions, correlations and in other problems.

It is easy to prove that $(x \mid q)^*$ is equal to $(q \mid x)$, i.e. the functions effecting reverse transformation. Actually, we shall multiply $(x \mid n)$ by $(x \mid q')^*$ and integrate with respect to x. Then, by virtue of the orthagonality and normalization of the function $(x \mid q)$ we obtain

$$\int (x \mid q')^* (x \mid n)\, dx = (q' \mid n),$$

hence the transformation function $(x \mid q')^* = (q' \mid x)$. From the representation it is clear that in the case of discrete spectra for example, for the quantities x and q, $|(x \mid q)|^2$ is simultaneously the probability of finding the variable q in the state x and the probability of finding x in the state q.

Let us cite examples of very well-known functions of quantum mechanics which play a major role as transformation functions. The eigen-function of the momentum operator in co-ordinate representation is a plane wave:

$$\Psi_p(r) = \frac{1}{(2\pi)^{3/2}} e^{ipr} \equiv (r \mid p).$$

And its complex conjugate $(r \mid p)^* = (p \mid r)$ is the eigen-function of r in the p-representation.

The eigen-function of the operator of angular momentum in a momentum representation has the form

$$Y_{lm}(\vartheta, \varphi) \equiv (\vartheta\varphi \mid lm).$$

ϑ, φ are the polar angles of momentum.

The following analysis is well-known:

$$\frac{1}{(2\pi)^{3/2}} e^{ipr} = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \frac{J_{l+\frac{1}{2}}(pr)}{\sqrt{pr}} i^l Y^*_{lm}(\Theta, \Phi)\, Y_{lm}(\vartheta, \varphi).$$

where Θ and Φ are the polar angles of the vector r, $J_{l+\frac{1}{2}}$ is a Bessel function. It can be written thus:

$$(r \mid p) = \sum_{l, m} (\Theta\Phi \mid lm)\,(r \mid p)_l\,(lm \mid \vartheta\varphi).$$

Here we obtain, in accordance with Section 21:

$$(lm \mid \Theta\Phi) = i^l Y^*_{lm}(\Theta, \Phi),$$

$$(r \mid p)_l = \frac{J_{l+\frac{1}{2}}(pr)}{\sqrt{pr}}.$$

These functions are orthogonal and normalized, and are important examples of transformation functions:

$$\sum_{l, m} (n' \mid lm)(lm \mid n) = (n' \mid n) = \delta(n - n').$$

Here n and n' are unit vectors, the direction of which is determined relative to the angles $\vartheta\varphi$ and $\vartheta'\varphi'$:

$$\int (l'm' \mid n)\, d\Omega_n\, (n \mid lm) = \delta_{l'l}\delta_{m'm},$$

$$\int (p' \mid r)_l\, r^2\, dr\, (r \mid p)_l = \frac{1}{p^2}\delta(p' - p).$$

We note that $\delta(p' - p) = \delta(n' - n)\frac{1}{p^2}\delta(p' - p)$.

Transformation of the matrices of operators from one representation into another is an obvious consequence of what has been stated:

$$(\eta' \mid F \mid \eta) = \sum_{\xi, \xi'} (\eta' \mid \xi')^* (\xi \mid P \mid \xi)(\xi \mid \eta),$$

i.e. the transformation is carried out by means of the same transformation functions. This, obviously, is also related to the operator $\hat{S}$.

Knowing the form of the S-matrix in a representation where constants of motion are taken for the variables ξ and ξ', and knowing the transformation functions from this representation into the representation corresponding to the condition of the experiment, $(\xi \mid \eta)$, it is easy to obtain the S-matrix in the required representation and consequently, also the relationship between it and the values obtained experimentally.

Section 23. Relationship between the S-Matrix and Effective Cross-section

As has already been mentioned, the square of the matrix element of the S-matrix defines the probability of detecting in the final process, one or other states of the system. Knowing this probability, it is possible to calculate the effective cross-section of the process. Up to the present we have considered the collision of particles in a very general form, implying under initial and final states, various states of the many particles.

We shall now consider the collision of two particles. Suppose that prior to collision (for $t \to -\infty$) particle I had a spin j_I, spin projection μ_I, particle II had a spin j_{II} and a spin projection μ_{II}; the particles were moving, not interacting, with a momentum p in a centre of mass system. We shall assign in the same way the so-called channel index* α - a quantity defining the type of particles I and

*We are adhering to the usual terminology in nuclear physics. Reaction channels are the different routes (in the sense of the properties of the reaction products) along which a reaction can proceed. In the concept of channel of course, is included not only the quantum numbers characterizing the internal state of the particles, but also the total spin of the particles and the orbital angular momentum.

II (mesons, neutrons, protons, etc.).

By the state f can be understood, for the present, any state of any number of particles which can arise as a result of the collision of particles I and II. The effective cross-section is defined as the ratio of the number of events of a given type occurring in unit time at one particular target, to the flux of the incident particles over unit surface area. According to this definition, it is required to find not the simple probability of obtaining the state f as a result of interaction, but the probability that this state occurs in unit time.

We have defined the S-matrix by the energy of the initial state and used the fact that energy is one of the constants of motion

$$(f|S|i)=(f_0|S^E|i_0)\,\delta(E_f-E_i), \tag{23.1}$$

where f_0 and i_0 denote the quantum numbers characterizing the final and initial states, excluding the energy E.

Squaring, we obtain

$$|(f\,S|i)|^2=|(f_0|S^E|i_0)|^2\cdot|\delta(E_f-E_i)|^2=$$
$$=\lim_{t\to\infty}|(f_0|S^E|i_0)|^2\,\delta(E_f-E_i)\int_{-t}^{t}\frac{1}{2\pi}e^{i(E_f-E_i)t}\,dt.$$

Integrating this expression with respect to all E_f, we find

$$\lim_{\Delta t\to\infty}\frac{1}{2\pi}|(f_0|S^E|i_0)|^2\Delta t.$$

From whence it follows that the probability of conversion in unit time is equal to

$$\frac{1}{2\pi}|(f_0|S^E|i_0)|^2. \tag{23.2}$$

Concerning the matrix $(f_0|S^E|i_0)$, one sometimes talks about the energy fixed on the surface. We shall find the flux of particles in the state $|i_0)$. We have defined the initial state by the set $|\alpha j_I\mu_I j_{II}\mu_{II}p)$. Since in this state the momentum has a definite value, then the flux, as is well-known [1] is equal to $v/(2\pi)^3$. Further, the state $|i_0)$ is defined, not by the quantum number p, but by the quantum member for

the energy of relative motion of the particles, E.

It is easy to show that the wave functions $|\alpha j_I\mu_I j_{II}\mu_{II}p)$ and $|i_0)$ of these states are connected by the simple relationship:

$$|i_0)=|\alpha j_I\mu_I j_{II}\mu_{II}p)\frac{p}{\sqrt{v}}\,^{*)}. \tag{23.3}$$

Hence we find the expression for the flux of particles in the state $|i_0)$:

$$\frac{p^2}{(2\pi)^3}. \tag{23.4}$$

Thus, the expression for effective cross-section takes the form:

$$\sigma_{f_0}=\frac{4\pi^2}{p^2}|(f_0|S^E|\alpha j_I\mu_I j_{II}\mu_{II}n)|^2. \tag{23.5}$$

σ_{f_0} can also be expressed by the matrix elements of the S-matrix in the representation where account is taken of the transformation of lm:

$$\sigma_{f_0}=\frac{4\pi^2}{p^2}\left|\sum_{lm}(f_0|S^E|\alpha j_I\mu_I j_{II}\mu_{II}lm)(lm|n)\right|^2. \tag{23.6}$$

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*We shall verify this relationship.
Let $\Psi_p\equiv\Psi_{pn}$ be the wave function, normalized in accordance with the condition:

$$\int\Psi^*_{pn}\Psi_{p'n'}\,d\tau=\delta(p-p')=\frac{1}{p^2}\delta(p-p')\,\delta(n-n')=$$
$$=\frac{1}{p^2\left(\frac{dp}{dE}\right)}\delta(E-E')\,\delta(n-n'),$$

here n is a single vector along the direction p. Equating this with the condition of normalization for Ψ_{En}

$$\int\Psi^*_{En}\Psi_{E'n'}\,d\tau=\delta(E-E')\,\delta(n'-n),$$

we find

$$\Psi_{En}=p\sqrt{\frac{dp}{dE}}\,\Psi_{pn}.$$

Noting that $\frac{dE}{dp}=v$, we obtain formula (23.3).

In accordance with the previous paragraph $(lm|n) = Y^*_{lm}(n)$. We shall select as the Z-axis, the direction of the incident beam, when

$$Y^*_{lm}(0,\ 0) = \sqrt{\frac{2l+1}{4\pi}}\,\delta_{-m0},$$

and expression (23.6) takes the form

$$\sigma_{f_0} = \pi\lambda^2 \left| \sum_l \sqrt{2l+1}\,(f_0|S^E|\alpha j_{\rm I}\mu_{\rm I} j_{\rm II}\mu_{\rm II} l0) \right|^2, \qquad (23.7)$$

where λ is the De Broglie wave length.

We shall show that the expression for cross-section agrees, in the limiting case, with the expression obtained in classical mechanics. Let the system possess such properties that a reaction takes place only if a particle has a definite orbital angular momentum $l = l'$. Let moreover, the reaction proceed with maximum intensity, i.e. the corresponding square of the matrix element of the S-matrix is equal to 1.

In this case, from (23.7) we find

$$\sigma_{l'} = \pi\lambda^2(2l'+1).$$

Let us derive this result from quasi-classical considerations. Let a parallel beam of particles impinge on the target. The angular momentum of the particles relative to the centre of the target is $p\rho$, where p is the momentum, and ρ the impact parameter, but, on the other hand, the angular momentum can take only the discrete values $\sqrt{l(l+1)}$, therefore for $l \gg 1$, the approximate equation will be fulfilled:

$$p\rho \approx l$$

or $\rho \approx l\lambda$, i.e. the particles with a given wave-length and orbital angular momentum strike at a fairly definite distance from the origin of the co-ordinates. We shall find the area of the anulus, in the plane perpendicular to the direction of the beam, on which particles with a given l will fall:

$$\sigma_l = \pi\rho^2_{l+1} - \pi\rho^2_l = \pi\lambda^2[(l+1)^2 - l^2] = \pi\lambda^2(2l+1).$$

Hence, it is obvious that the effective cross-section determined above, for large l converts into the cross-section defined in classical mechanics as the area of a circle in the plane perpendicular to the motion of the particles.

The expression derived in (23.7) is not the expression most generally used for the cross-section of any process operating as a result of collision of two particles. We have found the cross-section σ_f for the conversion of the state Ψ_i into the state Ψ_f, by taking the square of the matrix element of the operator $\hat{S}$. In the case of elastic scattering of particles, one is interested not merely in the probability of a system remaining in the original channel, but the probability, that as a result of interaction, the system is reconverted into the original channel. This implies that from the wave function Ψ_f, it is necessary to calculate the wave function Ψ_i:

$$\Psi'_f = \Psi_f - \Psi_i = (\hat{S} - \hat{I})\Psi_i,$$

where $\hat{I}$ is a unit matrix, and in order to calculate the cross-section of elastic scattering, the square of the matrix element of the operator $\hat{S} - \hat{I}$ must be found. The matrix $\hat{I}$, as is well-known (see [1]), remains unity in all representations. Therefore the general expression for cross-section (including also elastic processes) takes the form

$$\sigma_{f_0} = \pi\lambda^2 \left| \sum_l \sqrt{2l+1}\,[(f_0|\hat{S} - \hat{I}|\alpha j_{\rm I}\mu_{\rm I} j_{\rm II}\mu_{\rm II} l0)] \right|^2. \qquad (23.8)$$

If the beams of colliding particles are not polarized, then this expression must be summed over all values of the projections of the spins of the primary particles. This gives

$$\bar{\sigma}_{f_0} = \frac{1}{(2j_{\rm I}+1)(2j_{\rm II}+1)} \sum_{\mu_{\rm I}\mu_{\rm II}} \sigma_{f_0}. \qquad (23.9)$$

The general expression for cross-section through the matrix elements of the S-matrix in representations where transformations are taken for lm, instead of n has the form

$$\sigma_{f_0} = \frac{4\pi^2}{p^2} |(f_0|\hat{S} - \hat{I}|\alpha j_{\rm I}\mu_{\rm I} j_{\rm II}\mu_{\rm II} n)|^2. \qquad (23.10)$$

CHAPTER VII

APPLICATIONS OF THE GENERAL THEORY OF THE S-MATRIX

Section 24. Relationship between Effective Cross-sections of Elastic and Inelastic Processes

On the basis of formula (23.8) it is possible to draw a number of general conclusions on the properties of effective cross-sections.

1) Let us suppose that the interaction of the particles occurs only in a state with a specific orbital moment l_1, (i.e. a specification of interaction such that the elements of the S-matrix differ from zero only for $l = l_1$); then

$$\sigma^{l_1}_{f_0} = \pi \lambda^2 (2l_1 + 1) \left| (f_0 | S | \alpha j_I \mu_I \, j_{II} \mu_{II} l_1 0) \right|^2.$$

This value is called the partial cross-section.

From (19.1) it follows that $|(f_0|S|i_0)|^2 \leqslant 1$, i.e. the partial cross-section of any inelastic process cannot exceed $\pi\lambda^2(2l + 1)$. This agrees well with the result from classical mechanics considered previously. Thus the unitarity of the S-matrix (19.1) gives rise to the simple fact that the number of interactions producing the state f_0, cannot exceed the number of particles falling on the target.

2) No inelastic processes exist which are not accompanied simultaneously by the process of elastic scattering. In fact, even if one element of the S-matrix differs from zero, then by virtue of the property of unitarity, the absolute value of the diagonal elements of the S-matrix are less than unity, and this implies that the cross-section of elastic scattering, in accordance with (23.8), differs from zero.

3) Collision of spinless particles. From the general

results obtained above, it is possible to derive very simply the well-known expressions for the effective cross-section of scattering of spinless particles. In this case, using (20.1) and (20.2), we represent the S-matrix in the form

$$S^l \delta_{ll'} \delta_{mm'} \delta(E' - E). \tag{24.1}$$

From the requirement of unitarity, in the absence of inelastic processes it follows that

$$S^l S^{l*} = 1,$$

or

$$S^l = e^{2i\delta_l}. \tag{24.2}$$

Substituting this expression in formula (23.8), the final state in which is now characterized a definite moment l, we obtain the partial elastic scattering cross-section

$$\sigma^l = \pi\lambda^2 \left| \sqrt{2l+1}\,(e^{2i\delta_l} - 1) \right|^2 = 4\pi\lambda^2 (2l+1) \sin^2 \delta_l,$$

and we find the total cross-section by summation over all values of l:

$$\sigma = 4\pi\lambda^2 \sum_l (2l+1) \sin^2 \delta_l, \tag{24.3}$$

here δ_l is the phase shift.

In order to obtain the differential scattering cross-section, it is necessary to transform the S-matrix into a representation of the angles θ and φ:

$$d\sigma = \frac{\lambda^2}{4} \left| \sum_l (2l+1)(e^{2i\delta_l} - 1) P_l(\cos\theta) \right|^2 d\Omega\,^*. \tag{24.4}$$

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*The element of the solid angle arises as usual in expressions associated with quantum numbers having a continuous spectrum. If the wave functions are normalized by a δ-function, then the square of the modulus of the wave function determines the probability density; the probability is obtained by multiplying the square of the modulus of the wave function by the differential of the spectrum. For example, if $\Psi(x)$ is the wave function in co-ordinate presentation, then the probability of having a co-ordinate x is written down as $|\Psi(x)|^2 dx$. If it is a function for n particles in the momentum presentation, then in order to obtain the probability, it is necessary to multiply the square of

Integration of (24.4) with respect to angle, as is well-known, gives (24.3). In deriving formulae (24.3) and (24.4), it is obvious that we have used only the general properties of the S-matrix**.

We shall now consider the case when inelastic processes are also possible as a result of collision of spinless particles. In this case, it is no longer possible to derive from the condition of unitarity, a simple expression for the S-matrix, and under l in formula (24.1) should be understood the total anguler momentum of the secondary particles. For elastic scattering l is, as before, the orbital angular momentum. Therefore, the cross-section for elastic scatter-is obtained from (24.4) by replacing $e^{i\delta_l}$ with the complex number S^l having a modulus less than unity

$$d\sigma_s = \frac{\lambda^2}{4}\left|\sum_l (2l+1)(S^l-1)P_l(\cos\theta)\right|^2 d\Omega.$$

Integrating with respect to angle, we obtain the total cross-section of elastic scattering

$$\sigma_s = \pi\lambda^2 \sum_l (2l+1)|S^l-1|^2. \qquad (24.5)$$

We shall find the total cross-section for all inelastic processes by means of formula (23.8), choosing for f_0 a set of quantum numbers including the total angular momentum of the particles emitted:

$$\sigma_r = \pi\lambda^2 \sum_{n,l} (2l+1)|(n|S^l|0)|^2.$$

Under the summation over all values n, we understand as usual the summation over every discrete variable (including the enumeration of all types of reaction), and integration over every continuous variable.

the modulus of the function by the product of the differentials of the momenta of the particles: $dp_1\,dp_2\,dp_3\ldots dp_n$. This product is generally called "the element of the phase space". **Moreover here, of course, the assumption has been made that for $t\to\pm\infty$, the particles can be considered as free. For scattering potentials, falling off to infinity as $1/r$ and more slowly, special consideration is generally necessary.

Using the condition of unitarity

$$\sum_n |(n|S^l|0)|^2 + |(0|S^l|0)|^2 = 1,\ (0|S^l|0) \equiv S^l;$$

hence

$$\sigma_r = \pi\lambda^2 \sum_l (2l+1)(1-|S^l|^2). \qquad (24.6)$$

Let $S^l = B_l e^{i\delta_l}$. Then the partial elastic scattering cross-section can be written in the form

$$\frac{\sigma_s^l}{\pi\lambda^2(2l+1)} = 1 - 2B_l\cos\delta_l + B_l^2,$$

and the partial inelastic scattering cross-section in the form

$$\frac{\sigma_r^l}{\pi\lambda^2(2l+1)} = 1 - B_l^2.$$

From these expressions, it is particularly obvious that inelastic processes are accompanied at all times by elastic processes. For a given value of cross-section of elastic scattering, there is a maximum possible value for the cross-section of inelastic processes σ_r^{max}, and when $\sigma_r = \sigma_r^{max}$, σ_s is also equal to σ_r^{max}; the maximum possible cross-section for elastic scattering $\sigma_s^{max} = 4\sigma_r^{max}$.

4) We shall consider the limiting case, when the inelastic processes play a very great part. That is, let there be a great many channels, and since all matrix elements of the S-matrix are of approximately a single order of magnitude, and since $\sum_{f_0}|(f_0|S|i_0)|^2 = 1$, then every every element $|(f_0|S|i_0)| \ll 1$. The collision of mesons and nucleons at high energies is a typical example*:

$$\pi + N \rightarrow \begin{cases} \pi + N' \\ 2\pi + N' \\ 3\pi + N' \\ \cdot\ \cdot\ \cdot \end{cases}$$

The cross-section of each individual inelastic process is very small. However, in accordance with (24.6), the sum of

*With certain modifications these representations are widely used in nuclear physics.

the cross-section of all the inelastic processes is approximately equal to

$$\sigma_r \approx \sum_l \pi\lambda^2 (2l+1).$$

For summation with respect to the state f_0, we have used the relationship $\sum_{f_0 \neq i_0} |(f_0|S|i_0)|^2 = 1 - |(i_0|S|i_0)|^2$ and we have neglected $|(i_0|S|i_0)|^2$ in comparison with 1. We shall make a similar approximation for the cross-section of elastic scattering (24.5):

$$\sigma_s = \sum_l \pi\lambda^2 (2l+1).$$

The expression used for the matrix operator $\hat{S}-\hat{I}$ is not correct for all l. As mentioned previously, for large values of l, it can be considered quasi-classical, and these values of l correspond to large impact parameters $\rho = \lambda l$. If the radius of interaction of the colliding particles is equal to R (this, obviously, is a certain effective value, but not necessarily fixed rigidly), then the maximum value of l, for which interaction is still possible, is equal to $l_0 \approx \frac{R}{\lambda}$. This implies that for $l > l_0 \approx \frac{R}{\lambda}$ the particles do not interact and $\hat{S} = \hat{I}$. This is, summation with respect to l goes from zero to l_0. Hence, it is easy to show that $\sigma_r = \sigma_s = \pi R^2$. Putting it another way, the cross-section of elastic scattering will be very large - equal to the total cross-section of all the inelastic processes. Our result bears, at first sight, a paradoxical character, since the total cross-section of interaction is equal to $2\pi R^2$. This paradox is explained if the angular distribution of the elastically scattered particles is considered. For this it is necessary to transform the operator $\hat{S} - \hat{I} \approx -\hat{I}$ into a representation of angular scattering. The transformation function, as mentioned above, has the form:

$$Y_{lm}(\theta, \varphi) = (\theta\varphi \,|\, lm),$$

$$\sum_{l'm'} I_{l'lm'm} (l'm' \,|\, \theta\varphi) = Y_{lm}(\theta, \varphi). \qquad (24.7)$$

Let us substitute (24.7) in the general formula for the scattering cross-section (23.8); as a result we obtain

$$d\sigma = \pi\lambda^2 \left| \sum_{l=0}^{l_0 = \frac{R}{\lambda}} \frac{2l+1}{\sqrt{4\pi}} P_l(\cos\theta) \right|^2 d\Omega. \qquad (24.8)$$

It can be shown (see [2] p.495) that the summation in (24.8) is equal to

$$\frac{J_1\left(\frac{R}{\lambda}\theta\right)}{\left(\frac{\lambda}{R}\right)\theta},$$

where J_1 is a Bessel function of the first order. Thus, in this case, when inelastic processes are large (intensive aborption of particles from the beam takes place), the angular distribution of elastically - scattered particles has the form:

$$\frac{d\sigma}{d\Omega} = \pi R^2 \left| \frac{J_1\left(\frac{R}{\lambda}\theta\right)}{\theta} \right|^2. \qquad (24.9)$$

But $J_1\left(\frac{R}{\lambda}\theta\right)$ essentially differs from zero only for $\frac{R}{\lambda}\theta \sim 1$, or for $\theta \sim \frac{\lambda}{R}$. Hence the explanation of the paradox follows: for conversion to classical mechanics, the relationship $\sigma_t = \sigma_r + \sigma_s = 2\pi R^2$ holds but the particles are elastically scattered through very small angles $\sim \frac{\lambda}{R}$, consideration of which requires the application of quantum mechanics. The criterion of applicability of classical mechanics consists of precisely the fact that it should be possible to neglect the wave length of the particle in comparison with the dimensions of the system. But this implies that σ_s should be neglected, and σ_t becomes

$$\sigma_t = \pi\lambda^2 \sum_l (2l+1) = \pi R^2.$$

Expression (24.9) agrees exactly with the expression for Frauenhofer diffraction by a perfectly black sphere. This agreement is not accidental but is a consequence of the wave nature of particles. In connection with this, by optical analogy, the elastic scattering of particles caused by the presence of inelastic processes is called diffraction scattering. As is evident from what has been stated, the angular distribution is characteristically sharp, the half-width is determined by the ratio between the De Broglie wave length and the effective radius of interaction of the colliding particles.

5) <u>Optical Theorem</u>. In the example presented, we have used the property of unitarity of the S-matrix to show the close connexion between elastic and inelastic processes. We shall prove that between these processes there exists an accurate relationship, independent of those assumptions

which we have used for discussion of our example. This relationship is also a consequence of the fundamental property of the unitarity of the S-matrix.

We have seen above that $\hat{S}-\hat{I}$ is the operator, the squares of the matrix elements of which determine the cross-section of various processes. In place of the operator $\hat{S}-\hat{I}$, the operator $\hat{R}$ is frequently introduced, defined as $i\hat{R}=\hat{S}-\hat{I}$. $\hat{R}$ and $\hat{S}-\hat{I}$ differ only by an unimportant phase factor. Let us write down the property of unitarity of the S-matrix through the operator $\hat{R}$:

$$\hat{S}\hat{S}^{+}=(\hat{I}+i\hat{R})(\hat{I}-i\hat{R}^{+})=\hat{I}, \qquad (24.10)$$

whence

$$i\hat{R}-i\hat{R}^{+}+\hat{R}\hat{R}^{+}=0. \qquad (24.11)$$

Let us consider the matrix element of the operator equation (24.11), when the wave functions are the functions $\Psi=|\alpha\mu_1\mu_2 \boldsymbol{n}E)\frac{2\pi}{p}$. We have,

$$(\Psi', \hat{R}\Psi)=\frac{2\pi}{p'}(\boldsymbol{n}'\mu_1'\mu_2'\alpha'\,|\,\hat{R}\,|\,\boldsymbol{n}\mu_{\rm I}\mu_{\rm II}\alpha)\frac{2\pi}{p}\delta(E'-E). \qquad (24.12)$$

As can be seen from formula (23.10), the square of the modulus of the expression

$$\frac{2\pi}{p'}(\boldsymbol{n}'\mu_1'\mu_2'\alpha'\,|\,\hat{R}\,|\,\boldsymbol{n}\mu_{\rm I}\mu_{\rm II}\alpha). \qquad (24.12')$$

occurring in (24.12) gives the differential cross-section of elastic scattering. This value is called the amplitude of elastic scattering. The matrix element of the third term in (24.11) can be written down as

$$(\Psi', \hat{R}\hat{R}^{+}\Psi)=$$
$$=\sum_N(\boldsymbol{n}'\mu_1'\mu_2'\alpha'|\hat{R}|N)(N\,|\,\hat{R}^{+}\,|\,\boldsymbol{n}\mu_{\rm I}\mu_{\rm II}\alpha)\frac{(2\pi)^2}{p'p}\delta(E'-E). \qquad (24.13)$$

$\sum\limits_N$, as usual signifies the sum over every discrete variable and intergration over every continuous variable, which can occur as a result of collision of particles I and II. Substitution of (24.12) and (24.13) in (24.11) gives a system of integral equations for the amplitudes of the processes. The existence of such a system, as was seen from the introduction, follows only from the requirement of unitarity of the S-matrix. These integral equations have a particularly simple form when, by virtue of the Law of Conservation of energy, only a process of elastic scattering is possible. It is easy to show, that by means of the system of integral equations derived, it is possible to determine the phases of the scattering amplitudes, if their moduli are known. Thus, by measuring the effective cross-section, we find the squares of the moduli of the scattering amplitudes and then, by means of the integral equations, their phases. In practice, for elastic scattering of spinless particles we find

$$i\left[(\boldsymbol{n}'|\hat{R}^{+}|\boldsymbol{n})\frac{2\pi}{p}-(\boldsymbol{n}'\,|\,\hat{R}\,|\,\boldsymbol{n})\frac{2\pi}{p}\right]\frac{2\pi}{p'}=\int\frac{(2\pi)^2}{pp'}(\boldsymbol{n}'\,|\,\hat{R}\,|\,\boldsymbol{n}'')\,d\Omega''(\boldsymbol{n}''\,|\,\hat{R}^{+}\,|\,\boldsymbol{n})$$

or, introducing the symbol f for the elastic scattering amplitude, we obtain,

$$\frac{4\pi}{p}\operatorname{Im} f[(\boldsymbol{n}', \boldsymbol{n})]=\int f[(\boldsymbol{n}', \boldsymbol{n}'')]\,d\Omega'' f^{*}[(\boldsymbol{n}'', \boldsymbol{n})], \qquad (24.14)$$

where $(\boldsymbol{n}', \boldsymbol{n})$ is the scalar product of the singl vectors.

Relationships of the type of (24.14) were used, for example, in [8] for analysing a complete set of experiments* on the elastic collision of nucleons with nucleons. We shall consider only the particular case when the integral equations reduce to a simple relationship. Let $\boldsymbol{n}'=\boldsymbol{n}; \mu_1=\mu_{\rm I}$; $\mu_2=\mu_{\rm II}$ $\alpha'=\alpha$. Then the expression (24.12') will be the amplitude for forward scattering without change of projections of the spins (we shall denote it by $f(0)$), and (24.13) is reduced to the following simple expression:

$$\delta(E'-E)\,\sigma,$$

where σ is the total cross-section of all processes which can occur as a result of collision of two particles with given projections of spin μ_1 and μ_2. On the basis of (24.14) we find

$$\frac{4\pi}{p}\operatorname{Im} f(0)=\sigma. \qquad (24.15)$$

Or, expressing it another way, the imaginary portion of the forward scattering amplitude is proportional to the total effective cross-section.

Let us sum the left and right portions of (24.14) with

*See below.

respect to the spin projections of the colliding particles. Then we obtain

$$\frac{4\pi}{p}\,\mathrm{Sp\,Im}\,f(0)=(2j_{\mathrm{I}}+1)(2j_{\mathrm{II}}+1)\bar{\sigma}, \qquad (24.16)$$

where $\bar{\sigma}$, in accordance with (23.9) is the total cross-section for collision of non-polarized particles. The relationships (24.15) and (24.16) are the so-called optical theorem, and they are frequently used for theoretical consideration of the various processes of collision, and for the analysis of experimental data. Consider for example the process of scattering of photons in a Coulomb field. From the point of view of quantum electro-dynamics, the process is of a high order of e^2, however, the fact emerges that in a Coulomb field a fairly intense pair formation occurs (inelastic process), a marked scattering of the photons will occur (the so-called Delbrück scattering). Whereupon, by means of (24.14), knowing the cross-section for the pair formation, it is easy to calculate the lower limit of the forward scattering cross-section. An important application of the optical theorem is found in the analysis of dispersion ratios [7]. It can also be used for making more precise phase analysis of the various scattering processes, particularly in those cases when it is difficult to measure the cross-section of scattering through small angles.

From the optical theorem it is possible, without drawing upon the model dependent representations used previously, to obtain the basic shape of the angular distribution of elastically scattered particles for high energies.

Let us consider a simple case, the scattering of spinless particles; then, from (24.14) it follows that [34],

$$|f(0)|^2 \geqslant \left(\frac{p\sigma_t}{4\pi}\right)^2,$$

but this implies that the angular distribution of elastically scattered particles at high energies is sharply forward, and is focussed within the solid angle:

$$\Delta\Omega=\pi\theta^2 \leqslant \left(\frac{4\pi}{p\sigma_t}\right)^2\sigma_s.$$

Section 25. Relationship between Effective Cross-sections of Direct and Inverse Reactions

Let us consider the two reactions:

$$\mathrm{I}+\mathrm{II} \rightleftarrows 1+2.$$

The cross-section of the direct reaction, in accordance with (23.10), is written

$$\frac{d\sigma_{if}}{d\Omega_f}=\frac{4\pi^2}{p_i^2}\,|(-\boldsymbol{n}_f\mu_1\,\mu_2|S^E|\boldsymbol{n}_i\,\mu_{\mathrm{I}}\,\mu_{\mathrm{II}})|^2. \qquad (25.1)$$

The minus sign for the unit vector $\boldsymbol{n}_f$ denotes that in the state f, the particles are moving away from the original system of co-ordinates. The cross-section of the inverse reaction has the form:

$$\frac{d\sigma_{fi}}{d\Omega_i}=\frac{4\pi^2}{p_f^2}\,|(-\boldsymbol{n}_i\,\mu_{\mathrm{I}}\,\mu_{\mathrm{II}}|S^E|\boldsymbol{n}_f\,\mu_1\,\mu_2)|^2. \qquad (25.2)$$

Equating (25.1) and (25.2) with (21.15) we see that there is no simple relationship between the cross-section of the direct reaction and that of the inverse reaction in which the spins of the initial and final state are orientated in an opposite direction relative to the orientation of the spins in the direct reaction. It is then necessary for us to sum the cross-section with respect to the spin projections in the final state, and to sum over the projections in the initial state; this gives

$$\overline{\frac{d\sigma_{if}}{d\Omega_f}}=\frac{4\pi^2}{p_i^2}\,\frac{1}{(2j_{\mathrm{I}}+1)(2j_{\mathrm{II}}+1)}\times\sum_{\mu_{\mathrm{I}}\mu_{\mathrm{II}}\mu_1\mu_2}|(-\boldsymbol{n}_f\mu_1\mu_2|S^E|\boldsymbol{n}_i\mu_{\mathrm{I}}\mu_{\mathrm{II}})|^2$$

$$\overline{\frac{d\sigma_{fi}}{d\Omega_i}}=\frac{4\pi^2}{p_f^2}\,\frac{1}{(2j_1+1)(2j_2+1)}\times\sum_{\mu_{\mathrm{I}}\mu_{\mathrm{II}}\mu_1\mu_2}|(-\boldsymbol{n}_i\mu_{\mathrm{I}}\mu_{\mathrm{II}}|S^E|\boldsymbol{n}_f\mu_1\mu_2)|^2.$$

If now, we use relationship (21.15) and make the summation over all the spin projections, then we obtain the equation

$$\frac{\overline{d\sigma_{if}}}{p_f^2 d\Omega_f}(2j_{\mathrm{I}}+1)(2j_{\mathrm{II}}+1)=\frac{\overline{d\sigma_{fi}}}{p_i^2 d\Omega_i}(2j_1+1)(2j_2+1). \qquad (25.3)$$

As was evident from the derivation of the last equation, comparison of the cross-sections of the direct and inverse processes can only be made for one and the same energy E. Furthermore, it is necessary to remember that (25.3) is accurate only for the case when the colliding particles are not polarized. In the general case there is a more complex relationship between the direct and inverse processes, determined by formula (21.15). For reactions with polarized beams of particles it is also possible to derive a series of useful relationships between the cross-sections of the direct and inverse reactions.

Equation (25.3) is frequently described as the relationship of detailed balance. This description is inaccurate, since in classical physics by detailed balance is usually understood the equation for the probability of the direct and inverse processes, but this equation, as was pointed out above, does not hold. (The difference between a reciprocal and a detailed equation is discussed in [6].)

Let us consider examples of the application of relationship (25.3).

1) Determination of the spin of the π-meson. The spin of the π-meson was established with greater certainty by a study of the reaction

$$p+p \rightleftarrows d+\pi^{+}. \tag{25.4}$$

Application to it of relationship (25.3) gives

$$\frac{\dfrac{\overline{d\sigma}_{if}}{d\Omega_f}}{\dfrac{\overline{d\sigma}_{fi}}{d\Omega_i}}=\frac{3}{2}\frac{p_f^2}{p_i^2}(2s+1). \tag{25.5}$$

Here the factor $\frac{1}{2}$ is introduced, taking into account the identity of the two protons; s is the spin of the π-meson.

From formula (23.5) it is obvious that the cases $s=0$ and $s=1$ give a ratio of the cross-sections differing between themselves by a factor of 3. Thus, even the somewhat rough measurements of the differential cross-section of the reaction (25.4) for only one angle makes possible the determination of the fact that the spin of the π-meson is equal to zero.

In view of its generality, relationship (25.3) can also be used for determining the spins of other particles. Relationships between the cross-sections of direct and inverse reactions are frequently used also for reactions involving photons. In this case, however, it is necessary

to modify them somewhat*. Actually, although the spin of a photon is also equal to 1, the number of its different projections is equal not to $2j+1=3$, but only 2. (only two different polarizations of a photon are possible). Hence, it follows that modification of (25.3) is of a trivial nature: the factor $(2j+1)$ is replaced by 2 for a photon.

2) Of great fundamental significance is the measurement of the cross-section of the photoproduction of π-mesons by neutrons:

$$\gamma+n \to p+\pi^{-}. \tag{25.6}$$

however, neutron targets do not exist, and this process has to be studied by means of a somewhat complex interpretation of the process $\gamma+d \to 2p+\pi^{-}$. But obviously, in place of process (25.6), it is possible to study the inverse of this process, and determine the cross-section of the direct process by using relationship (25.3), taking into account the photon polarizations:

$$\frac{\overline{d\sigma}_{\gamma\pi}}{d\Omega_\pi}=\frac{\overline{d\sigma}_{\pi\gamma}}{d\Omega_\gamma}\frac{p_\pi^2}{p_\gamma^2}\frac{1}{2}.$$

3) In theoretical calculations of the photodisintegration of a deuteron,

$$\gamma+d \to p+n \tag{25.7}$$

it sometimes proves to be convenient to calculate the cross-section of the inverse process, and subsequently by means of (25.3) to obtain the cross-section of process (25.7).

In this case (25.3) is written

$$\frac{\overline{d\sigma}_{d,\,pn}}{d\Omega_p}=\frac{\overline{d\sigma}_{pn,\,d}}{d\Omega_\gamma}\frac{2}{3}\frac{p_p^2}{p_\gamma^2}.$$

*The question concerning reactions involving photons will follow below in more detail.

CHAPTER VIII

COLLISION OF PARTICLES POSSESSING SPIN

Section 26. Statement of the Problem. Examples. Determination of the Parameters of the S-Matrix

As indicated previously, from the general properties of the matrix of scattering, it is possible to obtain information concerning the cross-sections of interaction of spinless particles. In practice, the unitarity of the S-matrix and the conservation laws, allow a description of a process in terms of phase shifts. The specific character of an interaction is manifested in the magnitude of the phase shifts and their dependence upon the energies. For example, in the case when the radius of interaction is comparable with the wave length of the particle, the cross-section is described with great accuracy by the smallest number of phase shifts (this statement immediately follows from the conversion to classical mechanics made previously).

Thus, by studying the properties of particles and their interactions which are subject to investigation (theoretical and experimental), these parameters can be found. By carrying out phase shift analyses of experimental data it is possible to determine the properties of the processes which are related to the general laws of nature, and also the properties which are associated with a specific collision of particles of a specific kind.

Is it always possible to carry out phase shift analysis? What are the special features of collision of particles possessing spin?. These questions are the subject of discussion in the present chapter. Let us consider for a start elastic collision of particles with spins:

$$\mathrm{I}+\mathrm{II}\rightarrow 1+2. \qquad (26.1)$$

The difference in this case from elastic scattering of spinless particles is that the state of the system must be described not only the direction of motion of the particles (the angles θ, φ), but also by the values of the spin projections of the particles, μ. The S-matrix now has the form:

$$(\theta'\varphi'\mu_1\mu_2|S^E|\theta\varphi\mu_{\mathrm{I}}\mu_{\mathrm{II}})\,\delta(E'-E), \qquad (26.2)$$

and for spinless particles it had the form

$$(\theta'\varphi'|S^E|\theta\varphi)\,\delta(E'-E). \qquad (26.3)$$

In the case of spinless particles we replaced the variables θ, φ, by the variables l and m, and applied the conservation laws (20.2). Then the S-matrix had the form

$$S^l\delta_{l'l}\delta_{m'm}\delta(E'-E). \qquad (26.4)$$

The conversion from expression (26.3) to (26.4) was achieved by means of the transformation function $(\theta\varphi|lm)=Y_{lm}(\theta,\varphi)$

$$(\theta'\varphi'|S^E|\theta\varphi)=\sum_{lm}Y_{lm}(\theta',\varphi')S^lY_{lm}(\theta,\varphi)=\sum_l\frac{2l+1}{4\pi}P_l(\cos\omega)S^l,$$

where ω is the angle between the directions (θ,φ) and (θ',φ'), S^l are complex parameters, dependent on the energies. The application of unitarity allowed the introduction of a single effective parameter - the phase shift.

In the case of particles with spin, the application of the conservation laws gives

$$S^J\delta_{J'J}\delta_{M'M}\delta(E'-E), \qquad (26.5)$$

but S^J is now no longer a number but a matrix. In fact, we have assigned the four quantum numbers $\theta, \varphi, \mu_1, \mu_2$ to the state (not taking into account the energies), but the laws allow the possibility of using only the two quantum numbers J and M. Consequently, for the two remaining quantum numbers S is the matrix, and its dependence on these numbers is determined by the specific interaction*. Hence it is

*The number of quantum numbers required to determine a state is determined by the number of degrees of freedom of the system and obviously, does not depend upon the representation. Thus, for example, for spinless particles there evident, that in the case of a particle with spin, it is impossible, generally speaking, to simplify the expression for cross-section, as in the case of scattering of spinless particles (introduction of phase shifts). In particular cases, however, the introduction of phase shifts still appears to be feasible. Let us consider the scattering of spinless particles by particles with a spin of $\frac{1}{2}$ (for example, scattering of π -mesons by nucleons, or neutrons by He^4). The state is characterized by assigning the numbers (E, l, m, μ), where μ may assume two values: $\pm 1/2$. If we include the orbital angular momentum l and the spin of the particle in the total angular momentum of the system J and its projection M in place of the quantum numbers m and μ, then our system will be characterized by the numbers (E, l, J, M). Where, from the law of addition of angular momenta $J = l \pm \frac{1}{2}$. Taking into account the laws of conservation of angular momentum and of the energy, the S-matrix has the form:

$$S^J \delta_{J'J} \delta_{M'M} \delta(E' - E),$$

where S^J is a double-series matrix:

$$\begin{pmatrix} \left(J+\frac{1}{2}\,|\,S^J\,|\,J+\frac{1}{2}\right) & \left(J+\frac{1}{2}\,|\,S^J\,|\,J-\frac{1}{2}\right) \\ \left(J-\frac{1}{2}\,|\,S^J\,|\,J+\frac{1}{2}\right) & \left(J-\frac{1}{2}\,|\,S^J\,|\,J-\frac{1}{2}\right) \end{pmatrix}. \qquad (26.6)$$

Let us study one further law - the law of conservation of parity (see [2] 29):

$$\pi_i = (-1)^l \Pi_{\rm I} \Pi_{\rm II} = (-1)^{l'} \Pi_1 \Pi_2 = \pi_f,$$

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are three numbers (E, θ, φ), or (E, l, m) and so forth, but for two particles with spin there are 5 numbers $(E, \theta, \varphi, \mu_1, \mu_2)$, or (E, l, m, μ_1, μ_2). Should the spins of the particles be linked by a single total spin, then in place of μ_1 and μ_2, two new quantum numbers are introduced - the total spin s and its projection μ. As a result of this, the state will be assigned the numbers (E, l, m, s, μ). It is possible to link the spin with the orbital angular momentum by introducing instead of the projections m and μ, the quantum numbers J - the total angular momentum, and M, its projection: (E, J, M, l, s) is the new set of five quantum numbers. In the text, the discussion is concerned for example, with the quantum numbers l and s.

where π_i is the total parity of the initial state and π_f is the total parity of the final state, and $\Pi_{\rm I,\,II}$ and $\Pi_{1,2}$ are the intrinsic parities of the particles*: thus, in our case $l' = l + 2n$, where $n = 0, 1, 2, \ldots$ But, by the law of conservation of total angular momentum, l' can be different from l by not more than 1; consequently, the non-diagonal term in (26.6) is equal to zero, and the S-matrix of the diagonal term is:

$$S_l^J \delta_{J'J} \delta_{M'M} \delta_{l'l} \delta(E' - E).$$

The requirement of unitarity gives

$$S_l^{J*} S_l^J = 1 \quad \text{and} \quad S_l^J = e^{2i\delta_l^J},$$

i.e. in the case when one of the particles is spinless, and the other particle has a spin of $\frac{1}{2}$, the concept of phase shift retains its normal meaning. However, the phase shift depends not only on l but also on J.

Let us consider another important example - elastic scattering of particles with a spin of $\frac{1}{2}$ (for example, scattering of neutrons by protons).

The system is characterized by the quantum numbers (E, l, m, μ_1, μ_2). Let us introduce the quantum numbers for the constants of motion J and M and the quantum number for the spin of the particles s (it can assume the values 0 and 1).

$$(J'M'E's'l'\,|\,S\,|\,JMEsl) = (s'l'\,|\,S^J\,|\,sl)\,\delta_{J'J}\delta_{M'M}\delta(E' - E).$$

In accordance with the law of conservation of angular momentum

$$l = \begin{cases} J-1 \\ J \\ J+1 \end{cases} \quad \text{for} \quad s = 1 \quad \text{and} \quad l = J \quad \text{for} \quad s = 0.$$

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* We are occupied in the study of nuclear reactions, and in principle the question is one concerning strong interactions. Violations of the law of conservation of parity are detected only in weak interactions. In strong interactions parity is conserved.

Taking into account the law of conservation of parity gives the selection rule $l' = l + 2n$, where n is an integer. Thus only transitions between $l = J+1$ and $l = J-1$ and transitions without variation of l. are allowed. For a given total angular momentum and a given parity, the state of the S-matrix has the form

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

where the indices 1 and 2 refer respectively to the states $(l = J-1;\ s = 1)$ and $(l = J+1;\ s = 1)$, or the states $(l = J;\ s = 0)$ and $(l = J;\ s = 1)$. The matrix elements $S_{\mu\nu}$ are, of course, complex parameters and must be expressed through the real parameters $r_{\mu\nu}$ and $\delta_{\mu\nu}$:

$$S_{\mu\nu} = r_{\mu\nu} e^{i\delta_{\mu\nu}}.$$

The condition for unitarity, taking into account the theorem of reciprocity $(S_{\mu\nu} = S_{\nu\mu})$ can be written in the form of the matrix equation

$$SS^{+} = \begin{pmatrix} r_{11}e^{i\delta_{11}} & r_{12}e^{i\delta_{12}} \\ r_{12}e^{i\delta_{12}} & r_{22}e^{i\delta_{22}} \end{pmatrix} \begin{pmatrix} r_{11}e^{-i\delta_{11}} & r_{12}e^{-i\delta_{12}} \\ r_{12}e^{-i\delta_{12}} & r_{22}e^{-i\delta_{22}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (26.7)$$

This equation is a short version of a system of equations which gives the possibility of expressing one parameter through another. As a result of solution of this system, we find

$$S = \begin{pmatrix} \sqrt{1-r^2}\, e^{2i\delta} & ire^{i(\delta+\eta)} \\ ire^{i(\delta+\eta)} & \sqrt{1-r^2}\, e^{2i\eta} \end{pmatrix}. \qquad (26.8)$$

Here we have introduced three new parameters: δ, η and r; their relationship with the old parameters is obvious from a comparison of formulae (26.7) and (26.8).

Thus, in the case considered, the introduction of the phase shift concept in the usual sense has been shown to be impossible. The application of the general properties of the S-matrix has, however, sharply reduced the number of parameters which it is necessary to determine by experiment. Actually, even after application of the laws of conservation of angular momentum and of parity, we have obtained the S-matrix for a given J and parity, in the form of tables, which involve four complex parameters, i.e. eight independent real parameters. The application of unitarity and the theorem of reciprocity has led to the fact that for a given J and parity, the S-matrix is expressed entirely through three real parameters.

For the application of these results to elastic scattering of nucleons, it is necessary to take into consideration the identity of the particles (for np-scattering - isotopic invariancy). This results, as is easily seen, in singlet to triplet transitions $(s = 1 \rightleftarrows s = 0)$ being forbidden and thus to additional simplification of the structure of the scattering matrix. From the examples given, it is obvious that the higher the spin of the colliding particles, the more complex is the structure of the scattering matrix. The complexity of the structure of the scattering matrix is a simple consequence of the circumstance that we have at our disposal only a small number of limitations on the S-matrix, whereas with increase of spin the number of quantum numbers on which the S-matrix depends is increased.

A study of the structure of the scattering matrix has a great practical importance. As a result the S-matrix can be expressed through a small number of real parameters the magnitude of which is determined by the specific interaction of the particles. Subsequently, measurements are made of those interactions, from which it is possible to obtain complete information concerning the parameters of the S-matrix. As a result of this, a great many experiments are found to be superfluous - they give information which can be obtained from data from other experiments.

As an example of such an investigation the work [8] may be cited, where all possible polarization experiments for the elastic collision of nucleons were studied and it was pointed out which were the most advantageous and independent. In this work the S-matrix was expressed in terms of a different set of parameters to those we have used.

It should also be noted that all the considerations given are applicable, obviously, not only to elastic collisions but are very general. In section 34 we shall apply the theory given here for establishing the relationship between scattering processes, photoproduction of π -mesons and the Compton effect by a nucleon. This relationship arises from the same considerations, which were stated, if in the number of variables of the S-matrix μ and ν are included as independent indices of open channels.

It can be shown [14, 6] that a study of the fundamental properties of the S-matrix - unitarity and the theorem of reciprocity for the S-matrix, given in the form of a square table for N complex elements - reduces the number of independent real parameters from $2N^2$ to $\frac{1}{2}N(N+1)$ parameters.

Section 27. Vector Addition Coefficients

From the examples presented it is obvious that in the case of collision of particles possessing spin, as well as for the collision of spinless particles, it is possible to make a number of inferences concerning the structure of the S-matrix, arising out of the very general properties of space-time and from the laws of quantum mechanics.

Now we must ascertain the limits which these properties of the scattering matrix apply to the values observed - cross sections. For this, it is necessary to be able to transform the given S-matrix in one representation into another representation, in particular into that which applies to a particular experiment.

In the case of scattering of spinless particles, a single transformation function $(\theta\varphi|lm)$ was adequate - a spherical function. As is obvious from the examples presented above, we require transformation functions which transform a total representation (in which the S-matrix has the most simple form) into a representation of total angular momenta (spins of particles, orbital angular momenta). From this representation it is even possible to change to an angular representation and, by applying the general formulae for cross-section (23.8), to obtain the angular distribution and other characteristics.

Let the eigen-functions $\Psi_{j_1\mu_1}$ and $\Psi_{j_2\mu_2}$ of the operators of the angular momenta j_1 and j_2 be known, and it is required to find the eigen function Ψ_{JM} for the operator of their total angular momentum. The function Ψ_{JM} can obviously be represented in the form of an analysis with respect to the total system of the functions, consisting of the product of

$\Psi_{j_1m_1}$ and $\Psi_{j_2m_2}$.

$$\Psi_{JM} = \sum_{m_1m_2} C^{JM}_{j_1m_1j_2m_2}\Psi_{j_1m_1}\Psi_{j_2m_2}. \qquad (27.1)$$

If we know Ψ_{JM}, then $C^{JM}_{j_1m_1j_2m_2}$ should not be difficult to find by using an orthogonal function. In the Dirac notation assumed by us, equation (27.1) can be written as:

$$(x_1x_2|j_1j_2JM) = \sum_{m_1m_2}(j_1m_1j_2m_2|j_1j_2JM)(x_1|j_1m_1)(x_2|j_2m_2).$$

From this it is obvious that $(j_1m_1j_2m_2|j_1j_2JM)$ are precisely those transformation functions for which we are looking - they effect transformation from a representation of constituent angular momenta into a representation of total angular momentum. The singularity of these functions consists in the fact that they have as well as an index of states, an index of representation which are discrete quantities and assume a finite number of values. Consequently, the coefficients $(j_1m_1j_2m_2|j_1j_2JM)$ themselves represent elements of finite matrices. In spite of the simple physical significance of these coefficients, obtaining them in an explicit form involves somewhat complex mathematical calculations. The first general expression for these coefficients was given by Wigner [12]. For their derivation Wigner used the fancy mathematical device of Group Theory. Later Racah, in an important paper [13], showed that the coefficients can be derived by an algebraic route without recourse to Group Theory methods. The coefficients $C^{JM}_{j_1m_1j_2m_2}$ play a very great and ever increasing role in the various applications of quantum mechanics. A variety of titles and symbols exist for them. The most generally used title is "Clebsch-Gordon Coefficients" (according to the name of the authors of an important theorem in the theory of irreducible representations of group rotation). They are also called Wigner's coefficients and vector addition coefficients. We shall adhere to the latter title. We shall denote the symbols for vector addition coefficients as generally used in the literature:

$$(j_1m_1j_2m_2|JM);\quad (j_1j_2m_1m_2|j_1j_2JM);\quad C^{JM}_{j_1m_1j_2m_2};$$
$$C^{J}_{m_1m_2};\quad C^{j_1m_1j_2m_2}_{JM};\quad (-1)^{-J+M}\sqrt{2J+1}\,S_{j_1m_1j_2m_2J-M}.$$

The most commonly used are the first three. We shall use the first.

The reader, interested in the derivation of the general expressions for $(j_1m_1j_2m_2$ should acquaint himself with the appropriate separate works [12, 2], wherein is given an account of the theory of irreducible representations of group rotations. For narrow practical purposes (calculations of angular distributions, study of the properties of the scattering matrix, the problem of correlation of particles and the phenomenon of polarization of particles in nuclear reactions, etc.), it is sufficient to understand the physical significance of the coefficients $(j_1m_1j_2m_2|JM)$, as Dirac transformation functions, to know their general properties, and to know how to use the tables of these coefficients. We shall proceed to an account of this problem.

The formulae for the forward $(JM|j_1m_1j_2m_2)$ and the reverse transformation $(j_1m_1j_2m_2|JM)$, as we have already observed, are connected by the simple relationship

$$(JM|j_1m_1j_2m_2)^* = (j_1m_1j_2m_2|JM).$$

The representation generally used in the literature is such that the coefficients $(JM|j_1m_1j_2m_2)$ are real. This signifies that the coefficients of the direct and inverse transformations are simply equal. It is obvious that the physical results should be independent of the series into which we have summed the angular momenta j_1 and j_2, i.e. the coefficients $(JM|j_1m_1j_2m_2)$ and $(JM|j_2m_2j_1m_1)$ differ only by the phase factor (since the transformation coefficients, or what amounts to the same thing, the wave functions, describing one and the same state, can be distinguished only by the phase factors).

It can be shown that

$$(JM|j_1m_1j_2m_2) = (-1)^{j_1+j_2-J}(JM|j_2m_2j_1m_1). \tag{27.2}$$

In courses of quantum mechanics it is demonstrated that for addition of angular momenta, the quantum numbers J and M can assume the values:

$$J = |j_1 - j_2|,\quad |j_1 - j_2 + 1|,\ \ldots,\ j_1 + j_2,\quad M = m_1 + m_2. \tag{27.3}$$

From the latter equation it follows that the sum in (27.1), with respect to one of the indices, m_1, m_2, bears a formal character, since for given m_1 and m_2, M is already determined

by equation (27.3).

From the entry $(j_1M - m_2j_2m_2|JM)$ it is obvious that the vector addition coefficients can be represented in the form of a matrix, the rows of which are denoted by the number J, and the columns by the number m_2. The number of rows is equal to $2j+1$, where j is the minimum of the values j_1 and j_2. It is easy to see that the number of columns is equal to the same number. Thus, the number of vector addition coefficients is equal to $(2j+1)^2$. (In Appendix II tables of vector addition coefficients are given.) The operators $\hat{J}$, $\hat{j}_1$, and $\hat{j}_2$ are connected by the relationship $\hat{J} = \hat{j}_1 + \hat{j}_2$, but this relationship can also be written down thus:

$$\hat{j}_1 = \hat{J} - \hat{j}_2;\qquad \hat{j}_2 = \hat{J} - \hat{j}_1,$$

i.e. any one of the vectors entering into the triplet $(\hat{J}, \hat{j}_1, \hat{j}_2)$ (triplets of summed vectors are frequently called triads), can be represented as a resultant vector. The properties of the vector addition coefficients correspond to these relationships:

$$\begin{aligned}\frac{(-1)^{J-M}}{\sqrt{2J+1}}\cdot(JM|j_1m_1j_2m_2) &= \frac{(-1)^{j_1-m_1}}{\sqrt{2j_1+1}}(j_1m_1|JMj_2 - m_2) = \\ &= \frac{(-1)^{2j_1}(-1)^{j_2-m_2}}{\sqrt{2j_2+1}}(j_2m_2|j_1 - m_1JM).\end{aligned} \tag{27.4}$$

Moreover, the relationships occur:

$$\begin{aligned}(j_1m_1j_2m_2|JM) &= (-1)^{j_1+j_2-J}(j_1 - m_1j_2 - m_2|J - M) = \\ &= (-1)^{j_1+j_2-J}(j_2m_2j_1m_1|JM) = (j_2 - m_2j_1 - m_1|J - M).\end{aligned} \tag{27.5}$$

In addition to the properties enumerated, vector addition coefficients, as every other transformation function, have properties of orthogonality and normalization:

$$\left.\begin{aligned}&\sum_{m_1m_2}(J'M'|j_1m_1j_2m_2)(j_1m_1j_2m_2|JM) = \delta_{J'J}\delta_{M'M},\\ &\sum_{JM}(j_1m_1'j_2m_2'|JM)(JM|j_1m_1j_2m_2) = \delta_{m_1'm_1}\delta_{m_2'm_2},\end{aligned}\right\} \tag{27.6}$$

and, in accordance with the property of symmetry,

$$\sum_{m_1 M} (JM \mid j_1 m_1 j_2 m_2)(JM \mid j_1 m_1 j_2' m_2') = \frac{2J+1}{2j_2+1} \delta_{j_2 j_2'} \delta_{m_2 m_2'}. \qquad (27.7)$$

Finally, we shall introduce the explicit expression obtained by Wigner [12] for the vector addition coefficients for an arbitrary j_1 and j_2:

$$(JM \mid j_1 m_1 j_2 m_2) =$$
$$= \sqrt{\frac{(J+j_1-j_2)!\,(J-j_1+j_2)!\,(j_1+j_2-J)!\,(J+M)!\,(J-M)!\,(2J+1)}{(J+j_1+j_2+1)!\,(j_1-m_1)!\,(j_1+m_1)!\,(j_2-m_2)!\,(j_2+m_2)!}} \times$$
$$\times \sum_k \frac{(-1)^{k+j_2+m_2}(J+j_2+m_1-k)!\,(j_1-m_1+k)!}{(J-j_1+j_2-k)!\,(J+M-k)!\,k!\,(k+j_1+j_2-M)!}. \qquad (27.8)$$

Section 28. Some Examples

As an example of the use of the tables of properties of vector addition coefficients we shall consider the simplest, but important case of scattering of particles with a spin of $\frac{1}{2}$ by particles having a spin equal to zero.

In accordance with the general properties of the S-matrix (19.1) and (20.1), in the case being considered it will have the form

$$e^{2i\delta_l^J} \delta_{J'J} \delta_{M'M} \delta_{l'l} \delta(E'-E). \qquad (28.1)$$

In order that it should be possible to use formula (23.8) and to obtain the cross-section, it is necessary to transform the S-matrix from the representation $JMlE$ into the representation $\mu m l E$. The corresponding transformation functions $(JMsl \mid ls\mu m)$ is not the same as the vector addition coefficient of the vectors of the spin $\hat{s}$ and the orbital angular momentum $\hat{l}$ in the total angular momentum $\hat{J}$. Performing this transformation on (28.1) we obtain

$$\sum_{J'M'} e^{2i\delta_l^J} \delta_{J'J} \delta_{M'M} \delta_{l'l} \delta(E'-E)(J'M' \mid s\mu l m) =$$
$$= e^{2i\delta_l^J} \delta_{ll'} \delta(E'-E)(JM \mid s\mu l m). \qquad (28.2)$$

If we do not transform the final state to another variable, and we substitute (28.2) in (23.8), then we obtain the cross-section of collision of the particles with a fixed orbital angular momentum and its projection in the final state.

This cross-section is written down in the following manner:

$$\sigma_{l'\mu_1}^{JM} = \pi\lambda^2 (2l'+1) \left|1 - e^{2i\delta_{l'}^J}\right|^2 \left|(JM \mid s\mu_1 l'0)\right|^2. \qquad (28.3)$$

We shall sum (28.3) with respect to $l'M$ and also over the two possible spin directions of the initial particles to obtain the result for an unpolarized beam. The latter implies the operation $\frac{1}{2}\sum_{\mu_1}$. By virtue of the property (27.7) of vector addition coefficients, we obtain the result:

$$\sigma^J = \sum_l \pi\lambda^2 4 \sin^2 \delta_l^J (2J+1) \frac{1}{2}.$$

The total cross-section will have the form

$$\sigma = \sum_J \sigma^J = 4\pi\lambda^2 \sum_l \left[(l+1)\sin^2 \delta_l^{l+\frac{1}{2}} + l \sin^2 \delta_l^{l-\frac{1}{2}}\right]. \qquad (28.4)$$

If the phase shift is independent of the spin, then $\delta_l^{l+\frac{1}{2}} = \delta_l^{l-\frac{1}{2}}$ and (28.4) converts into the total cross-section of scattering of spinless particles.

We shall assume that the phase shift of the state $J = 3/2$, $l = 1$ passes through $\frac{\pi}{2}$ (in other words, this state resonates at three energies, for which we shall consider collision). The quantum numbers chosen correspond to resonance, or as it is otherwise known, the isobaric state of interaction of the π-mesons with nucleons. In this case, the partial cross-section attains its maximum value (geometrical limit):

$$8\pi\lambda^2.$$

If we wish to find the angular distributions, then the transformation function for the final state of transition into angular representation is required. For this, it is first necessary to convert to the representation $lms\mu$ in the left hand portion of the matrix equation, having used the transformation function $(lm\, s\mu \mid JM)$, and then, by means of the function $(\theta\varphi \mid lm)$ to find the S-matrix in the representation $\theta\varphi$.

We shall not carry out here the corresponding calculations, since we shall obtain below the general formulae for angular distributions for nuclear reactions.

If we are interested in the application of the complete relationships to the scattering of π-mesons, then it is necessary to include the isotopic spin in the number of variables. But, as is well-known, the laws of addition of isotopic spins are the same as the laws of addition of normal angular momenta. Therefore, a generalization of the theory to include the scattering of particles possessing an isotopic spin does not present any difficulty.

Isotopic spin is a constant of motion, consequently the S-matrix has the form:

$$S_l^{JT}\delta_{J'J}\delta_{l'l}\delta_{M'M}\delta_{T'T}\delta_{T_3'T_3}\delta(E'-E),$$

here T and T_3 respectively, are the total isotopic spin of the system and its projection. The requirement for unitarity gives

$$S_l^{JT}=e^{2i\eta_l^{JT}}.$$

For $l=0$, only the value $J=1/2$, is possible, for $l\neq 0$ two values are possible: $J=l\pm\frac{1}{2}$. The isotopic spin for a meson-nucleon system can take the values $3/2$ and $1/2$. Hence it is clear that S-wave scattering of mesons of all three charge states, by a neutron or a proton, is described by $\delta_0^{\frac{1}{2}\frac{1}{2}}$ and $\delta_0^{\frac{1}{2}\frac{3}{2}}$. Scattering in the conditions with $l\neq 0$ is described by four phase shifts

$$\delta_l^{l+\frac{1}{2}\,\frac{3}{2}},\quad \delta_l^{l+\frac{1}{2}\,\frac{1}{2}},\quad \delta_l^{l-\frac{1}{2}\,\frac{3}{2}},\quad \delta_l^{l-\frac{1}{2}\,\frac{1}{2}}.$$

All the transformations which we have carried out above hold. It is still necessary to make additional transformations of the S-matrix into a representation appropriate for an experiment. And indeed, experiments shows that a meson of a specific sign of charge (positive, negative, neutral) is scattered by a nucleon into a definite state of isotopic spin, i.e. in the experiment we have the state where the isotopic spins of both particles are $t=1$ and $\tau=\frac{1}{2}$, and their projections $t_3=\pm 1$ and 0, $\tau_3=\pm\frac{1}{2}$, and the properties of the S-matrix are known in the representation, where the sum of the isotopic spins and its projection are given. The transformation functions $\left(TT_3\,|\,1t_3\tfrac{1}{2}\tau_3\right)$, obviously, are the

same vector addition coefficients, because $\hat{T}=\hat{t}+\hat{\tau}$, and the laws of addition of isotopic spins conform exactly with the laws of addition of normal spins.

For the scattering π^+-mesons by protons, $\left(TT_3\,|\,1\;1\tfrac{1}{2}\tfrac{1}{2}\right)$ in accordance with the table in Appendix II is equal to

$$\begin{cases} 1 \text{ for } T=T_3=\frac{3}{2} \\ 0 \text{ for all remaining } T \text{ and } T_3. \end{cases}$$

Hence it is clear that the example considered previously (without taking into account isotopic spin applies completely to this case. The scattering amplitude for the process $\pi^-+p\rightarrow\pi^-+p$ is expressed in the following manner by the scattering amplitude in the states of isotopic spin $T=3/2$ and $1/2$:

$$(\pi^-p\alpha'\,|\,R\,|\,\pi^-p\alpha)=\sum_{T'TT_3'T}\left(1-1\tfrac{1}{2}\tfrac{1}{2}\,\middle|\,T'T_3'\right)\times$$
$$\times(\alpha'\,|\,R\,|\,\alpha)\,\delta_{T'T}\delta_{T_3'T_3}\left(TT_3\,|\,1-1\tfrac{1}{2}\tfrac{1}{2}\right).$$

Hence α' and denote all the remaining transformations except transformations of isotopic spin. By using the tables of vector addition coefficients, we find

$$(\pi^-p\alpha'\,|\,R\,|\,\pi^-p\alpha)=\frac{1}{3}\left(\alpha'\,\middle|\,R^{\frac{3}{2}}\,\middle|\,\alpha\right)+\frac{2}{3}\left(\alpha'\,\middle|\,R^{\frac{1}{2}}\,\middle|\,\alpha\right).$$

As an exercise we suggest the student expresses the total cross-sections and angular distributions of the reaction $\pi+p\rightarrow\begin{cases}p+\pi^-\\ p+\pi^0\end{cases}$ in terms of phase shift analysis. For the total cross-sections, the expression

$$\sigma(\pi^-\rightarrow\pi^-)=$$
$$=\frac{3\pi}{4k^2}\sum_{l=0}^{\infty}\left\{(l+1)\left[\sin^2\delta_{l^+}^{\frac{3}{2}}+2\sin^2\delta_{l^+}^{\frac{1}{2}}-\frac{2}{3}\sin^2\left(\delta_{l^+}^{\frac{3}{2}}-\delta_{l^+}^{\frac{1}{2}}\right)\right]+\right.$$
$$\left.+l\left[\sin^2\delta_{l^-}^{\frac{3}{2}}+2\sin^2\delta_{l^-}^{\frac{1}{2}}-\frac{2}{3}\sin^2\left(\delta_{l^-}^{\frac{3}{2}}-\delta_{l^-}^{\frac{1}{2}}\right)\right]\right\},$$

$$\sigma(\pi^-\rightarrow\pi^0)=$$
$$=\frac{8\pi}{9k^2}\sum_{l=0}^{\infty}\left\{(l+1)\sin^2\left(\delta_{l^+}^{\frac{3}{2}}-\delta_{l}^{\frac{1}{2}}\right)+l\sin^2\left(\delta_{l^-}^{\frac{3}{2}}-\delta_{l^-}^{\frac{1}{2}}\right)\right\}.$$

should be obtained. Here the upper sign of the phase shift denotes the isotopic spin and δ_{l+} and δ_{l-} denote the phase shifts corresponding to a given l, and to the angular momenta corresponding to $l+1/2$ and $l-1/2$. The formulae obtained for the angular distributions are too cumbersome for them to be introduced here.

Section 29. The Coefficients W, X, Z, Z_γ *

In the theory of the spectra of complex atoms, of angular correlations of particles resulting from disintegration, and of angular distributions for nuclear reactions, there arise unwieldly summations derived for certain vector addition coefficients. We shall see this when we obtain a general expression for the total and differential cross-sections for reactions of the type: $I+II \rightarrow 1+2$. In order to simplify the calculations and to obtain more compact expressions, Racah in a paper [13] concerning the theory of spectra introduced the coefficients W, subsequently known as Racah coefficients. These coefficients have also found extensive application in a number of other problems. In recent papers by Racah and other authors [13, 19, 27], a number of other coefficients were also introduced for a similar purpose.

The coefficients W, similar to the vector addition coefficients, may be defined correct to a factor as transformation functions permitting a transposition from one representation into another. But if vector addition coefficients arise in the problem associated with the addition of two operators of angular momentum $\hat{j}_1$ and $\hat{j}_2$, then the coefficients W arise in the problem of addition of three operators $\hat{j}_1, \hat{j}_2$ and $\hat{j}_3$. Let us find the wave function appearing as the eigen function of the operators $\hat{J}^2=(\hat{j}_1+\hat{j}_2+\hat{j})^2$ and $\hat{J}_z$. This problem is easily solved by means of vector addition coefficients, if the functions $(x_1|j_1m_1)$, $(x_2|j_2m_2)$ and $(x_3|j_3m_3)$ are known. The eigen function of the operators $\hat{J}_{12}=(\hat{j}_1+\hat{j}_2)^2$ and $\hat{J}_{12z}$ is written down thus:

$$(x_1x_2|J_{12}M_{12})= \\ =\sum_{m_1m_2}(x_1|j_1m_1)(x_2|j_2m_2)(j_1m_1j_2m_2|J_{12}M_{12}), \qquad (29.1)$$

*This paragraph was written by V.A. Petrun'kin.

and the corresponding function for the operators $\hat{J}^2$ and $\hat{J}_z$ can be obtained by the application of a similar formula to the functions $(x_1x_2|J_{12}M_{12})$ and $(x_3|J_3m_3)$ Hence we find

$$(x_1x_2x_3|J_{12}j_3JM)=\sum_{M_{12}m_3}(x_1x_2|J_{12}M_{12})(x_3|j_3m_3)(J_{12}M_{12}j_3m_3|JM)$$

or

$$(x_1x_2x_3|J_{12}j_3JM)= \\ =\sum_{m_1m_3}(x_1|j_1m_1)(x_2|j_2M-m_1-m_3)(x_3|j_3m_3)\times \\ \times(j_1m_1j_2M-m_1-m_3|J_{12}M-m_3)(J_{12}M-m_3j_3m_3|JM). \qquad (29.2)$$

Here we have substituted $(x_1x_2|J_{12}M_{12})$ in expression (29.1) and used the fact that the coefficient $(a\alpha b\beta|c\gamma)$ differed from zero only as a result of the condition that $\alpha+\beta=\gamma$. It should be possible to obtain the corresponding function for the operators $\hat{J}^2$ and $\hat{J}_z$, coupling the function $(x_i|j_im_i)$ in another sequence, for example, first of all the second with the third and the result with the first. In this case we obtain

$$(x_1x_2x_3|j_1J_{23}JM)=\sum_{m_1m_2}(x_1|j_1m_1)(x_2|j_2m_2)(x_3|j_3M-m_1-m_2)\times \\ \times(j_2m_2j_3M-m_1-m_2|J_{23}M-m_1)(j_1m_1J_{23}M-m_1|JM). \qquad (29.3)$$

Since the functions $(x_i|j_im_i)$ are orthonormal and represent a complete set, then the functions $(x_1x_2x_3|J_{12}j_3JM)$, $(x_1x_2x_3|j_1J_{23}JM)$ will also be orthonormal and will also represent a complete set. This follows from consideration of vector addition coefficients as transformation functions producing a conversion from one representation into another. In the case of (29.2) the state is characterized by the following six quantum numbers: $j_1, j_2, j_3, J_{12}, J, M$; in the case of (29.3) we have $j_1, j_2, j_3, J_{23}, J, M$. We shall resolve the function in (29.3) in terms of the function of (29.2):

$$(x_1x_2x_3|j_1J_{23}JM)=\sum_{J_{12}}(x_1x_2x_3|J_{12}j_3JM)(J_{12}j_3J|j_1J_{23}J). \qquad (29.4)$$

In order to obtain an explicit form of the coefficient $(J_{12}j_3J|j_1J_{23}J)$, we shall use the fact that the functions (29.2) are orthonormal. Multiplying the left side and the right side of the series (29.4) by the function $(x_1x_2x_3|J_{12}j_3JM)$ and integrating with respect to $x_1x_2x_3$, we obtain an expression for the coefficient required:

$$(J_{12}j_3J \mid j_1J_{23}J) = \sum_{mm_2} (j_1M - mj_2m_2 \mid J_{12}M - m + m_2) \times \\ \times (j_2m_2j_3m - m_2 \mid J_{23}m)(j_1M - mJ_{23}m \mid JM) \times \\ \times (J_{12}M - m + m_2j_3m - m_2 \mid JM). \quad (29.5)$$

From (29.4) it is not difficult to see that the expression $(J_{12}j_3J \mid j_1J_{23}J)$ is not the same as the wave function of the state $j_1J_{23}J)$ in the representation $(J_{12}j_3J|$ and consequently, according to Dirac, it can be used for transformation from the representation $(j_1J_{23}J|$ into the representation $(J_{12}j_3J|$. Indeed, relationship (29.4) was used by Racah for determination of the coefficients W. in his notation we have

$$(2e+1)^{\frac{1}{2}}(2f+1)^{\frac{1}{2}} W(abcd;\ ef) = (e_{ab}dc \mid af_{bd}c). \quad (29.6)$$

Racah evaluated the sum with respect to m, m_2 in (29.5) and obtained a general expression for W, but since it is so cumbersome it will not be given here. It need only be noted, that after summation the dependence upon M vanishes. This, in fact, has already been used earlier in equation (29.4). From the definition of W, it is obvious that they have significance only for integral and semi-integral values: a, b, c, d, e, f — with the following limitations:

a) The sum of each of the triplets of numbers enumerated below

$$(abe), \quad (edc), \quad (bdf), \quad (afc) \quad (29.7)$$

should be a whole number;

b) In order that the coefficient W should differ from zero, each of the triplets of numbers should satisfy the principle of the triangle, i.e. the length of any side of the triangle should be less or equal to the sum of the lengths of the other two sides.

The derivation of both limitations is clear from the definition of W. Racah's coefficients have a high degree of symmetry with respect to the permutation of their arguments.

The relationships shown below characterize the fundamental properties of symmetry of the Racah coefficients (these

properties are easily obtained, resulting from the general expression for W which we have omitted):

$$W(abcd;\ ef) = W(badc;\ ef) = W(cdab,\ ef) = \\ = W(acbd;\ fe) = (-1)^{e+f-a-d} W(ebcf;\ ad) = \\ = (-1)^{e+f-b-c} W(aefd;\ bc). \quad (29.8)$$

It is not difficult to obtain also a number of other properties of the coefficients W Since the coefficients $(e_{ab}dc \mid af_{bd}c)$ are transformation functions from one representation to another, the transformation corresponding to them should be unitary. This property is written down thus:

$$\sum_e (e_{ab}dc \mid af_{bd}c)(e_{ab}dc \mid ag_{bd}c) = \\ = \sum_e (2e+1)(2f+1)^{\frac{1}{2}}(2g+1)^{\frac{1}{2}} W(abcd;\ ef) \times \\ \times W(abcd;\ eg) = \delta_{fg}.$$

Using the orthonomality of the function $(x_i \mid j_i m_i)$ and substituting (29.2) and (29.3) in (29.4), the following expression can be obtained:

$$(a\alpha b\beta \mid e\,\alpha+\beta)(e\,\alpha+\beta\,d\delta \mid c\ \alpha+\beta+\delta) = \\ = \sum_f (2e+1)^{\frac{1}{2}}(2f+1)^{\frac{1}{2}}(b\beta d\delta \mid f\,\beta+\delta) \times \\ \times (a\alpha f\ \beta+\delta \mid c\ \alpha+\beta+\delta)\, W(abcd;\ ef). \quad (29.9)$$

This expression can be used to obtain the summation in terms of a single index of the multiplication of three vector addition coefficients:

$$\sum_\beta (a\alpha b\beta \mid e\ \alpha+\beta) \times \\ \times (e\ \alpha+\beta\,d\ \gamma-\alpha-\beta \mid c\gamma)(b\beta d\ \gamma-\alpha-\beta \mid f\ \gamma-\alpha) = \\ = (2e+1)^{\frac{1}{2}}(2f+1)^{\frac{1}{2}}(a\alpha f\ \gamma-\alpha \mid c\gamma)\, W(abcd;\ ef). \quad (29.9a)$$

Formula (29.9a) is obtained from expression (29.9) if the unitarity of vector addition coefficients is applied. This is the basic formula for simplification of the summation over multiplications of vector addition coefficients. In an actual case when $e = 0$, the value of the Racah coefficient can be obtained from the simple expression:

$$W(abcd;\ 0f) = (-1)^{b+c-f}(2b+1)^{-\frac{1}{2}}(2c+1)^{-\frac{1}{2}}\delta_{ab}\delta_{cd}. \quad (29.10)$$

This formula is obtained from the general expression for the coefficients W, which we shall not introduce. We shall not discuss here the methods of obtaining the recurrent formulae and tabulated Racah coefficients. In Appendix II calculated tables of W for various values of a, b, c, d, e, f are given.

In one of Racah's subsequent works, written jointly with Fano [16] and also in a paper [17], the coefficients X, were introduced. These coefficients are introduced in complete analagy with the coefficients W. Only in this case, they deal with the addition of four operators of angular momentum. The coefficients X, like W, is simply related by a certain function, effecting transformation from one representation to another:

$$[(2J_{12}+1)(2J_{34}+1)(2J_{13}+1)(2J_{24}+1)]^{\frac{1}{2}} X\begin{pmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{pmatrix} = (J_{12}J_{34}J \mid J_{13}J_{24}J). \tag{29.11}$$

Here $\hat{J}_{12}=\hat{j}_1+\hat{j}_2$, $\hat{J}_{34}=\hat{j}_3+\hat{j}_4$, $\hat{J}_{13}=\hat{j}_1+\hat{j}_3$, $\hat{J}_{24}=\hat{j}_2+\hat{j}_4$ and $\hat{J}_{12}+\hat{J}_{34}=\hat{J}$ or $\hat{J}_{13}+\hat{J}_{24}=\hat{J}$. The coefficient X is a function of ten arguments, being able to assume only integral or semi-integral values. As on the triads of numbers from (29.7) limits are imposed upon each of the triplets of the numbers

$$(j_1j_2J_{12}),\quad (j_3j_4J_{34}),\quad (j_1j_3J_{13}),\quad (j_2j_4J_{24}),\quad (J_{12}J_{34}J),\quad (J_{13}J_{24}J)$$

If the corresponding wave functions are written in the manner used in the previous case, then it is not difficult to find an explicit expression for the coefficient X. It will be represented by the sum of the products of six vector addition coefficients with respect to a magnetic quantum number. In future, a portion of the relationship and also the properties of the coefficients X will be given without proof. For details one can refer to the work [17]. The expression referred to above for the coefficient X can be transformed into a more simple form if Racah coefficients are used.

Ultimately we obtain

$$X\begin{pmatrix} a & b & e \\ c & d & e' \\ f & f' & g \end{pmatrix} = \sum_{\lambda}(2\lambda+1)W(fgbd;\ f'\lambda)W(egcd;\ e'\lambda)W(fcbe;\ a\lambda). \tag{29.12}$$

It is not difficult to show that the Racah coefficients are a particular case of the coefficients X. Actually, if we assume one of the six arguments b, c, d, e, f or g encountered twice on the right in (29.12) equal to 0, and we use the properties of symmetry for W (29.8) and formula (29.10), then we obtain a simple relationship between X and W. For example, if we assume $g=0$ and $e=e'$ and $f=f'$ respectively, then we obtain

$$X\begin{pmatrix} a & b & e \\ c & d & e \\ f & f & 0 \end{pmatrix} = (-1)^{e+f-a-d}(2e+1)^{-\frac{1}{2}}(2f+1)^{-\frac{1}{2}}W(abcd;\ ef). \tag{29.13}$$

On the basis of this, the coefficients X are sometimes called generalized Racah coefficients. The generalized basic Racah coefficient formula (29.9a) is very useful in its applications, and has the form

$$\sum_{m_1m_2\mu_1\mu_2}(j_1m_1l_1\mu_1 \mid s_1\sigma_1)\times(j_2m_2l_2\mu_2 \mid s_2\sigma_2)(j_1m_1j_2-m_2 \mid jm_j)(l_1\mu_1l_2-\mu_2 \mid lm_l) = (-1)^{s_2-j_2-l_2}[(2s_1+1)(2s_2+1)(2j+1)(2l+1)]^{\frac{1}{2}}\times \sum_{gm_g}(s_1\sigma_1s_2-\sigma_2 \mid gm_g)(jm_jlm_l \mid gm_g)X\begin{pmatrix} j_1 & j & j_2 \\ s_1 & g & s_2 \\ l_1 & l & l_2 \end{pmatrix}. \tag{29.14}$$

Let us enumerate some of the properties of the coefficients X, which can be obtained from the properties of the coefficients W.

(1) Transposition of rows and columns

$$X\begin{pmatrix} a & b & e \\ c & d & e' \\ f & f' & g \end{pmatrix} = X\begin{pmatrix} a & c & f \\ b & d & f' \\ e & e' & g \end{pmatrix}. \tag{29.15}$$

(2) Permutation of two rows or two columns

$$X\begin{pmatrix} a & b & e \\ c & d & e' \\ f & f' & g \end{pmatrix} = (-1)^{\sigma}X\begin{pmatrix} c & d & e' \\ a & b & e \\ f & f' & g \end{pmatrix} = (-1)^{\sigma}X\begin{pmatrix} f & f' & g \\ c & d & e' \\ a & b & e \end{pmatrix}, \tag{29.15a}$$

where $\sigma = a+b+c+d+e+e'+f+f'+g$ is equal to a whole number. Combining (29.15) and (29.15a) we can obtain 72 different permutations for nine arguments of X. For example, we can rewrite formula (29.12) in a more symmetrical form

$$X\begin{pmatrix} a & b & e \\ c & d & e' \\ f & f' & g \end{pmatrix} = (-1)^{\sigma} \sum_{\lambda} (2\lambda+1)\, W(bcef;\ \lambda a) \times \times W(bcf'e';\ \lambda d)\, W(efe'f';\ \lambda g). \qquad (29.16)$$

where the diagonal elements of X are the final arguments of three W coefficients. This formula is generally used as a standard formula for the relationship between X and W. Two further useful relationships can be written down:

$$\sum_{ff'} (2f+1)(2f'+1)\, X\begin{pmatrix} a & b & e \\ c & d & e' \\ f & f' & g \end{pmatrix} X\begin{pmatrix} a & b & e_1 \\ c & d & e_1' \\ f & f' & g \end{pmatrix} = [(2e+1)(2e'+1)]^{-1} \delta_{ee_1} \delta_{e'e_1'}$$

$$\sum_{ff'} (-1)^{e'-f'-2c+h'} (2f+1) \times \times (2f'+1)\, X\begin{pmatrix} a & b & e \\ c & d & e' \\ f & f' & g \end{pmatrix} X\begin{pmatrix} a & c & f \\ d & b & f' \\ h & h' & g \end{pmatrix} = X\begin{pmatrix} a & b & e \\ d & c & e' \\ h & h' & g \end{pmatrix}.$$

As has been shown in papers [18, 19], for the case of nuclear reactions it is more convenient in place of W to introduce the coefficients Z and Z_{γ}. The coefficients Z is defined by the following formula:

$$Z(abcd;\ ef) = i^{f-a+c} [(2a+1)(2b+1)(2c+1)(2d+1)]^{\frac{1}{2}} \times \times W(abcd;\ ef)\,(a0c0\,|\,f0). \qquad (29.17)$$

There is a simple formula for calculation of the coefficient $(a0c0\,|\,f0)$, which is given in Appendix II. Thus, evaluation of the coefficient Z reduces to the evaluation of the coefficient W corresponding to it. The coefficient $(a0c0\,|\,f0)$ is equal to zero when $a+c+f$ is equal to an odd number. Therefore, for Z we have the additional relationship

$$Z = 0, \text{ if } a+c+f \text{ is odd.}$$

Using formula (29.10) and also the properties of symmetry

(29.8), it is easy to obtain an expression for two actual cases:

$$Z(abcd;\ 0f) = \delta_{ab}\delta_{cd}(-1)^{2f} i^{f-a+c} [(2a+1)(2c+1)]^{\frac{1}{2}} (a0c0\,|\,f0). \qquad (29.18)$$

$$Z(abcd;\ e0) = \delta_{ac}\delta_{bd}(-1)^{b-e}(2b+1)^{\frac{1}{2}}. \qquad (29.18a)$$

Here we have used the formula for the coefficient $(a0b0\,|\,00)$

$$(a0b0\,|\,00) = \delta_{ab}(-1)^{a}(2a+1)^{-\frac{1}{2}}. \qquad (29.19)$$

As will be shown in sections 30 and 33, it is more convenient to define the coefficient Z without the factor i^{f-a+c}. But since in all major works and tables of the coefficient Z the definition in (29.17) is accepted, we have not deemed it an advantage to change it. We define the coefficient Z_{γ} thus:

$$Z_{\gamma}(abcd,\ ef) = [(2a+1)(2b+1)(2c+1)(2d+1)]^{\frac{1}{2}} \times \times W(abcd,\ ef)\,(a-1c1\,|\,f0). \qquad (29.20)$$

The formula is considerably simplified for the case $e=0$,

$$Z_{\gamma}(abcd,\ 0f) = (-1)^{b+c-f} [(2a+1)(2c+1)]^{\frac{1}{2}} (a-1c1\,|\,f0)\,\delta_{ab}\delta_{cd}. \qquad (29.21)$$

Our definition of Z_{γ} differs somewhat from the definition of the coefficients being used in similar cases by other authors.

Section 30. Angular Distributions in Nuclear Reactions
(Cases when the particles have a non vanishing rest mass)

The mathematical device described previously permits a general expression to be obtained for the differential cross-section of an arbitrary reaction of the type*:

$$\mathrm{I} + \mathrm{II} \rightarrow 1 + 2. \qquad (30.1)$$

*See papers [19, 22, 27].

In accordance with what was said previously, this expression distinguishes clearly between properties of cross-section associated with the general laws (laws of conservation, the general laws of quantum mechanics), and properties associated with a specific reaction (these properties will be distinguished by parameters of the type found in phase shift analysis).

In Section 28 we considered the simplest examples. Now we shall turn to the general case.

From (20.2) it follows that the S-matrix in the representation of constants of motion (in the CM-system) has the following form:

$$(s'l'\alpha' \,|\, S^J \,|\, sl\alpha)\,\delta_{J'J}\delta_{M'M}\delta(E'-E). \qquad (30.2)$$

where α and α' denote quantum numbers characterizing the nature and internal structure respectively, of the primary and secondary particles. For example, if we had applied our theory to the collision of hydrogen atoms, then the quantum numbers characterizing the state of the electrons in these atoms would occur in (30.2). In the following we shall not consider particular α and α', but only those interchangeable S-matrices about which we are able to give a definite opinion. In α we have included the quantum numbers of the spins of the particles, but not their projections. s and s' in (30.2) are the total spins of the channels, the vector sum of the spins for the incident particles is $\hat{s}=\hat{i}_{\mathrm{I}}+\hat{i}_{\mathrm{II}}$ and for the secondary particles is $\hat{s}'=\hat{i}_1+\hat{i}_2$. l and l' are the orbital angular momenta (in the CM-system) of the incident and secondary particles respectively.

Our aim is to obtain the differential cross-section of reaction (30.1). The relationship between the S-matrix and cross-section is given by the general formula (23.8). The differential cross-section $\frac{d\sigma}{d\Omega}$ is obtained from (23.8) if the angles of scattering θ and φ are included in the number of variables denoted by f_0. Let us convert (30.2) to these variables. For this, we shall first change to the variables $s'l'm'_sm'$ (m'_s is the quantum number for the projections of s', and m' the projections of l'). With this transformation function there will be, obviously, the vector addition coefficients $(J'M' \,|\, s'm'_sl'm')$. After this we convert to the variables θ and φ by means of the transformation function $(\theta\varphi \,|\, l'm') = Y_{l'm'}(\theta, \varphi)$.

In accordance with (23.8), we must take as the variables of the initial state $lm\mu_{\mathrm{I}}\mu_{\mathrm{II}}$. For this we transform (30.2) by means of $(JM \,|\, sm_slm)$ and after this by means of $(sm_s \,|\, i_{\mathrm{I}}\mu_{\mathrm{I}}i_{\mathrm{II}}\mu_{\mathrm{II}})$.

As a result of this we find that in (23.8) in place of $[(f_0 \,|\, S \,|\, l0i_{\mathrm{I}}\mu_{\mathrm{I}}i_{\mathrm{II}}M_{\mathrm{II}}) - I]$ it is necessary to substitute the expression

$$\sum_{JMl'm'sm_s} Y_{l'm'}(\theta, \varphi)\left(s'l'm'_sm \,|\, JM\right)(s'l'\alpha' \,|\, R^J \,|\, sl\alpha) \times$$
$$\times (JM \,|\, sm_sl0)(sm_s \,|\, i_{\mathrm{I}}\mu_{\mathrm{I}}i_{\mathrm{II}}\mu_{\mathrm{II}}). \qquad (30.3)$$

Our problem is now virtually solved; it is only necessary to transform and simplify the final result. In the first place, our formula is related to the case, rarely realized in practice, when the incident particles and the target particles have a strictly defined orientation of spins in space; in the final state we have also fixed a definite value for the quantities s' and m'_s. The case encountered most frequently is that in which neither the incident particles nor the target particles are polarized, where the polarized particles in the final state are of no interest. Thus, after substitution of (30.3) in (23.8), we must normalize to the initial state and sum over the final states of the spins of the particles. Using the orthogonality of the coefficients $(sm_s \,|\, j_{\mathrm{I}}\mu_{\mathrm{I}}j_{\mathrm{II}}\mu_{\mathrm{II}})$, we find

$$\frac{d\sigma_{\alpha'\alpha}}{d\Omega} = \frac{\lambda_\alpha^2}{(2j_{\mathrm{I}}+1)(2j_{\mathrm{II}}+1)} \sum (l'_1s'\alpha' \,|\, R^{J_1} \,|\, l_1s\alpha)^* \times$$
$$\times (l'_2s'\alpha' \,|\, R^{J_2} \,|\, l_2s\alpha)\,K(J_1l'_1l_1;\; J_2l'_2l_2;\; s's;\; \theta). \qquad (30.4)$$

The summation is made over all $J_1l_1l'_1J_2l_2l'_2ss'$.

From the quantity K we have excluded the factors determined only by the kinematics. For a specific collision (a particular channel), the matrix elements of the operator $\hat{R}$ are determined.

The expression for K:

$$K(J_1l'_1l_1;\; J_2l'_2l_2;\; s's;\; \theta) =$$
$$= (2l_1+1)^{\frac{1}{2}}(2l_2+1)^{\frac{1}{2}}\pi \sum_{m_1m_2m_sm'_s} (l0sm_s \,|\, J_1M_1) \times$$
$$\times (l_20sm_s \,|\, J_2M_2)\left(l_1m_1s'm'_s \,|\, J_1M_1\right)\left(l'_2m'_2s'm'_s \,|\, J_2M_2\right) \times$$
$$\times Y^*_{l_1m_1}(\theta, \varphi)\,Y_{l_2m_2}(\theta, \varphi)$$

can be considerably simplified by reducing it to a combination of Z coefficients and Legendre polynominals. Using the formula

$$Y^{*}_{l_1 m_1}(\theta, \varphi)\, Y_{l_2 m_2}(\theta, \varphi) = \sum_{LM} \left[\frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)} \right]^{\frac{1}{2}} (L0 \mid l_1 0 l_2 0) \times (LM \mid l_1 - m_1 l_2 m_2)(-1)^{m_1} Y_{LM}(\theta, \varphi). \qquad (30.5)$$

we obtain K in the form of a sum of the products of vector addition coefficients. These summations can be transformed by means of formula (29.9a). The summations over m'_s, m'_1 and m'_2 are incorporated in the Racah summation. Subsequently the summations over m_s, M_1 and M_2 are similarly incorporated in the Racah summation. This Racah summation is proportional to the coefficient $(LM \mid l_1 0 l_2 0)$, whence it follows that $M = 0$. The final result, after simple transformation, is written in the form

$$\frac{d\sigma}{d\Omega} = \frac{\lambda^2}{(2j_{\mathrm{I}}+1)(2j_{\mathrm{II}}+1)} \sum_{L=0}^{\infty} B_L P_L(\cos\theta), \qquad (30.6)$$

where $P_L(\cos\theta)$ is a Legendre polynominal, and

$$B_L = \sum \frac{(-1)^{s'-s}}{4} Z(l_1 J_1 l_2 J_2;\, sL)\, Z(l'_1 J_1 l'_2 J_2;\, sL) \times \left\{ i^{l'_2 - l'_1 + l_1 - l_2} \operatorname{Re}\left[(l'_1 s' \alpha' \mid R^{J_1} \mid l_1 s \alpha)^{*} (l'_2 s' \alpha' \mid R^{J_2} \mid l_2 s \alpha) \right] \right\}. \qquad (30.7)$$

The summation is over $J_1 J_2 l_1 l_2 l'_1 l'_2 s$ and s'. It is not difficult to see that expression (30.7) satisfies the theorem of reciprocity*.

Summation over every quantum number is formally extended from 0 to ∞. However, only the summation over one of the numbers is unlimited (for example, with respect to J_1). The remainder are limited by the rules of choice of the W coefficients, through which the Z coefficients are expressed. We note that all the terms in summation (30.7) are real.

*Formula (30.7) differs from the similar formula of the paper [19] by the phase factor $i^{l'_2 - l'_1 + l_1 - l_2}$. The difference is associated with the fact that the S-matrix of the paper [19] does not satisfy the requirement of invariancy under time reversal (see Section 21 and [21]).

For practical calculations, the following expression for B_L is more convenient:

$$B_L = \sum_{ss'} \frac{(-1)^{s'-s}}{4} \sum_{J=0}^{\infty} \sum_{l=|J-s|}^{J+s} \sum_{l'=|J-s'|}^{J+s'} Z(lJlJ;\, sL) \times Z(l'Jl'J;\, s'L) \left| (l's'\alpha' \mid R^J \mid ls\alpha) \right|^2 +$$
$$+ \sum_{s=|j_1-j_2|}^{j_1+j_2} \sum_{s'=|j_{\mathrm{I}}-j_{\mathrm{II}}|}^{j_{\mathrm{I}}+j_{\mathrm{II}}} \frac{(-1)^{s'-s}}{4} \sum_{J_1=0}^{\infty} \sum_{l_1=|J_1-s|}^{J_1+s} \sum_{l'_1=|J_1-s'|}^{J_1+s'} \left\{ \sum_{J_2=J_1+1}^{\infty} \sum_{l_2=|J_2-s|}^{J_2+s} \sum_{l'_2=|J_2-s'|}^{J_2+s'} Z(l_1 J_1 l_2 J_2;\, sL) \times \right.$$
$$\times Z(l'_1 J_1 l'_2 J_2;\, s'L) \operatorname{Re}[\quad] + \sum_{l_2=l_1+1}^{J_1+s} \sum_{l'_2=|J_1-s'|}^{J_1+s'} Z(l_1 J_1 l_2 J_1;\, sL) \times$$
$$\times Z(l'_1 J_1 l'_2 J_1;\, s'L) \operatorname{Re}[J_2 = J_1] + \sum_{l'_2=l'_1+1}^{J_1+s'} Z(l_1 J_1 l_1 J_1;\, sL) \times$$
$$\left. \times Z(l'_1 J_1 l'_2 J_1;\, s'L) \operatorname{Re}[J_2 = J_1,\, l_2 = l_1] \right\}.$$

The expression enclosed by the bracket in (30.7) should be substituted in the square brackets. In this expression, each term is encountered only once. As well as the limitation of the extent of the individual summations, the number of terms in (30.7) are also decreased by the following conditions: $(l_1 + l_2 - L)$ and $(l'_1 + l'_2 - L)$ are even numbers: $(l_1 + l'_1)$ and $(l_2 + l'_2)$ are even (odd) numbers, if the channels α and α' have the same (opposite) parity. The parity of a channel is defined as the product of the inherent parity of primary and secondary particles. Moreover, there are limitations on , well-known under the so-called theorem concerning the complexity of angular distributions [28], which is easily obtained from the properties of the Racah coefficients (see Section 29): $L \leqslant 2l^{\max}, 2J^{\max}, 2l'^{\max}$. In the right hand portion of these inequalities, which should be satisfied simultaneously, are the maximum values of the angular momenta participating in the process.

It is usually necessary to use formula (30.6) either when small values for the orbital angular momenta are involved, or when the reaction goes via a definite state with respect to the total angular momentum, or when the colliding particles have small spins: 0, $\frac{1}{2}$ or 1. In these cases, the formulae deduced are considerably simplified. The limiting number of Z-coefficients entering into expressions obtained in this manner, are taken from the tables given in Appendix II.

One of the most important applications of formula (30.6) consists in carrying out a generalized phase shift analysis of experimental data. This analysis consists of the following. The experimental angular distribution is expressed in terms of Legendre polynominals. From a comparison of these with formula (30.6), the matrix elements of the S-matrix $(l's'\alpha'|S|ls\alpha)$ are determined, i.e. those parameters in the angular distribution which are determined not by the kinematics but by a special feature of the process are found.

In the general case, such an analysis is ambiguous. As was indicated above, the quantities $(l's'\alpha'|S|ls\alpha)$ are matrices containing a large number of real parameters, and a study of the angular distribution gives a number of equations which is considerably less than the number of unknown parameters. Generally speaking, for a complete determination of the parameters, investigations of the collision of polarized particles are necessary, giving supplementary equations for determination of the parameters.

Even in the case of elastic scattering of particles with a spin of $\frac{1}{2}$ by particles having zero spin, analysis of the angular distribution does not give the complete information concerning the parameters of the S-matrix - the phase shifts. In this case, as was indicated in section 26, the S-matrix has the form:

$$e^{2i\delta_l^J}\delta_{l'l}\delta_{s's}, \qquad (30.8)$$

where $l=J\pm 1/2$, whence it follows that for given J_1 J_2 and L, Z-coefficients of four types will occur in the coefficients of B_L:

$$Z\left(J_1-\tfrac{1}{2}\,J_1J_2-\tfrac{1}{2}\,J_2;\ \tfrac{1}{2}\,L\right)=Z_{11},$$
$$Z\left(J_1+\tfrac{1}{2}\,J_1J_2-\tfrac{1}{2}\,J_2;\ \tfrac{1}{2}\,L\right)=Z_{12},$$
$$Z\left(J_1-\tfrac{1}{2}\,J_1J_2+\tfrac{1}{2}\,J_2;\ \tfrac{1}{2}\,L\right)=Z_{21},$$
$$Z\left(J_1+\tfrac{1}{2}\,J_1J_2+\tfrac{1}{2}\,J_2;\ \tfrac{1}{2}\,L\right)=Z_{22}.$$

In accordance with Section 29, Z_{11} and Z_{22} differ from zero when J_1+J_2-L is even, and Z_{12} and Z_{21} differ from zero when J_1+J_2-L is odd. Hence it follows that in the formula for angular distribution, only the combinations $Z_{11}Z_{11}$, $Z_{22}Z_{22}$, $Z_{11}Z_{22}$ and $Z_{12}Z_{21}$. enter. But these coefficients have two properties of symmetry: $Z_{11}=Z_{22}$ and $Z_{12}=-Z_{21}$.

From these results it is not difficult to see that the value of B_L is not decreased if the substitution is made of $\delta^J_{J+\frac{1}{2}}\rightleftarrows\delta^J_{J-\frac{1}{2}}$ for all J simultaneously.

In other words, the determination of the coefficients of B_L from analysis of experimental data, does not give a determination of the phase shifts. This ambiguity [32] is essential for analysis of data with respect to scattering of π-mesons by nucleons; it has a simple physical significance [33].

By means of (30.6) and (30.8), it is easy to show that as a result of meson-nucleon scattering, there is also an ambiguity of phase shift analysis, associated with the choice of the sign of the phase shifts - change of all signs does not decrease the values of the B_L coefficients. These ambiguities can be eliminated by analysis of experiments in which the polarization of the nucleons after the meson scattering has occurred is measured, and also by analysis of the interference between the Coulomb and nuclear scattering.

We shall leave the student, as an exercise, to obtain from the general formula (30.6), the differential scattering cross-section for spinless particles and the scattering of particles with a spin of $\frac{1}{2}$ by particles with a spin of 0, and to express these cross-sections via the phase shifts, for the lowest values of the orbital angular momentum $l=0,1$. The latter result is important for phase shift analysis of the scattering of π-mesons by nucleons. We also propose to prove that the angular distribution of the particles is spherically symmetrical, if the reaction goes only via the state of total angular momentum 0 or $\frac{1}{2}$.

CHAPTER IX

POLARIZATION OF PARTICLES IN NUCLEAR REACTIONS

Section 31. General Formulae

In connexion with the development of experimental techniques, the greatest value is derived from experiments with polarized particles. The reason is that these experiments measure the dependence of the scattering matrix on the variables characterizing new degrees of freedom. Experiments with variation of only angular distributions give only data summed over these variables. A number of relations, arising from polarization and particularly from reactions with polarized particles, can be obtained which arise purely from the general properties of the S-matrix as discussed by us in the first chapter. Consequently, the relations discussed below are perfectly general, not dependent on the nature of the particles participating in the reaction and the details of their interaction. We shall confine ourselves here to a discussion of the polarization of particles as a result of collision of a beam of non-polarized incident particles with non-polarized target particles. Consideration of the general case* for a reaction with polarized particles, and also correlations as a result of multiple processes, requires a greater complexity range of threatment and is beyond the scope of this book.

Polarization, according to definition, is the mean value of the operator of spin:

$$(\Psi, \hat{j}\Psi) = \bar{J} \qquad (31.1)$$

*See papers [18 and 24]. A discussion of correlation resulting from the decay of particles is given in greater detail in [27].

This, obviously, is a vector quantity.

We shall interest ourselves in the polarized particles arising as a result of collision processes. We will consider the polarization of particles characterized by the index 1. We will denote by Ψ the wave function of the particles formed as a result of the reaction. This wave function in accordance with section 23 has the form

$$\Psi'_f = \sqrt{2\pi}\sum_l \sqrt{\pi\lambda\!\!\!^{-2}(2l+1)}\,(f_0|R|\alpha j_{\rm I}\mu_{\rm I} j_{\rm II}\mu_{\rm II} l0). \qquad (31.2)$$

If, into this expression formula (30.3) is substituted in place of the matrix element $\hat{R}$, then wave functions are obtained with indices of representation $\vartheta\varphi s' m'_s$ indices, on which the operators act. Hence, it can be seen that for calculation of the polarization (31.1), an explicit form of the operator $\hat{j}_1$ is required in the representation $s'm'_s$:

$$(s'_1 m'_{s_1}|\hat{j}_1|s'_2 m'_{s_2}). \qquad (31.3)$$

Having calculated the matrix (31.3), and then having obtained the product of (31.1), i.e. having summed over $s'_1 s'_2 m'_{s_1} m'_{s_2}$ and integrated over all angles, we obtain the mean value of the polarization vector (the value obtained by us will be of importance for the cross-section). This is the so-called total polarization. If integration over the angles θ, φ, is not carried out in the product of (31.1), then the so-called differential polarization which is of most interest is obtained. It is important for the differential cross-section; and is the mean value of the spin operator of the particles omitted in unit time into the solid angle $d\Omega$ if the flux of the incident particles is unity. For differential polarization we shall adopt the notation $\frac{\overline{dP}}{d\Omega}$.

Frequently, the concept of relative polarization is introduced:

$$f = \frac{1}{j_1}\,\frac{\frac{\overline{dP}}{d\Omega}}{\frac{d\sigma}{d\Omega}},$$

where $\frac{d\sigma}{d\Omega}$ is the differential cross-section.

Let us pass on to calculation of the matrix (31.3). In [2] (Section 25), we introduced a general expression for the of the operators of the projection of the angular momentum. These expressions are obtained only from commutation conditions and therefore they also apply to the spin operator:

$$(\mu|\hat{j}_x + i\hat{j}_y|\mu') = \sqrt{(j+\mu)(j-\mu+1)}\,\delta_{\mu'\mu-1},$$
$$(\mu|\hat{j}_x - i\hat{j}_y|\mu') = \sqrt{(j-\mu)(j+\mu+1)}\,\delta_{\mu'-1\mu},$$
$$(\mu|\hat{j}_z|\mu') = \mu\delta_{\mu'\mu}.$$

Comparing these expressions with the tables of vector addition coefficients for $j=1$, the following connexion between them and the general formula can be seen*:

$$(\mu'|\hat{j}_\nu|\mu) = \sqrt{(j+1)j}\,(j\mu'|1\nu j\mu), \quad \nu = 0, \pm 1, \qquad (31.4)$$

where

$$\hat{j}_0 = \hat{j}_z, \quad \hat{j}_{\pm 1} = \mp\frac{1}{\sqrt{2}}(\hat{j}_x \pm i\hat{j}_y).$$

Let us transform this matrix into the representation we require:

$$(s_1'm_{s_1}'|\hat{j}_\nu|s_2'm_{s_2}') = \sum_{\mu_1'\mu_1}(s_1'm_{s_1}'|j_1\mu_1'j_2\mu_2)\sqrt{j_1(j_1+1)}(j_1\mu_1'|1\nu j_1\mu_1)(j_1\mu_1 j_2\mu_2|s_2m_{s_2}').$$

Simplifying this expression by means of the Racah formula (29.9a), we find

$$\sqrt{j_1(j_1+1)}\left[\frac{1}{3}(2s_1'+1)(2s_2'+1)(2j_1+1)\right]^{\frac{1}{2}}\times$$
$$\times(-1)^{j_2-j_1+m_{s_1}'-\nu}(s_1'-m_{s_1}'s_2m_{s_2}'|1-\nu)W(j_1s_1'j_1s_2';j_21). \qquad (31.5)$$

---

*Formula (31.4) obtained here, is a particular case, well-known in the application of Group Theory as the Wigner-Eckart theorem:

$$(j_2\mu_2|\hat{T}_q^k|j_1\mu_1) = (j_2|T_q|j_1)(j_2\mu_2|qkj_1\mu_1).$$

Here $\hat{T}_q^k$ is an operator which as a result of reversal of the system of co-ordinates transforms like $(\theta\varphi)$, $(j_2|T_q|j_1)$ is the reduced matrix element, independent of the projections of the angular momenta (see [2], p.409).

Let us multiply the right hand side of (31.5) by Ψ_f' and the left hand side by $\Psi_f'^*$ and sum over $s_1'm_{s_1}'s_2'm_{s_2}'$. In addition, since particles I and II are not polarized, we shall normalize with respect to the spin of the initial state

$$dP_\nu = \frac{\pi\lambda^2}{(2j_I+1)(2j_{II}+1)}\sqrt{j_1(j_1+1)}\frac{(2j_1+1)^{\frac{1}{2}}}{\sqrt{3}}\times$$
$$\times\sum[(2l_1+1)(2l_2+1)]^{\frac{1}{2}}[(l's_1'\alpha'|R^{J_1}|ls\alpha)]^*[(l_2's_2'\alpha'|R^{J_2}|l_2s\alpha)]\times$$
$$\times\sum_{m_s m_{s_1}' m_{s_2}'}(l_10sm_s|J_1m_s)(l_20sm_s|J_2m_s)(l_1'm_1's_1'm_{s_1}'|J_1m_s)\times$$
$$\times(l_2'm_2's_2'm_{s_2}'|J_2m_s)Y_{l_1m_1}^*Y_{l_2m_2}\times$$
$$\times(-1)^{j_2-j_1+m_{s_1}'-\nu}[(2s_1'+1)(2s_2'+1)]^{\frac{1}{2}}(s_1'-m_{s_1}'s_2'm_{s_2}'|1-\nu)\times$$
$$\times W(j_1s_1'j_1s_2';j_2s)\,d\Omega. \qquad (31.6)$$

The summation over magnetic quantum numbers can be simplified in a manner similar to that used for angular distributions. Apart from the application of Racah's formula, it is advantageous to introduce, by means of formula (29.16), the sum of the products of the three Racah coefficients in the coefficient X determined in Section 29. The details of these somewhat unwieldy but simple expressions is contained in the paper [29], which we have used considerably in this paragraph. The final formula for the cyclic projections of the polarization vector has the form:

$$\frac{dP_\nu}{d\Omega} = \sum A_L(L01\nu|L\nu)Y_{L\nu}(\theta,\varphi).$$

The complex expression for A_L is independent of ν.

From the tables of vector addition coefficients, it follows that

$$\left.\begin{aligned}
\frac{dP_z}{d\Omega} &= \frac{dP_0}{d\Omega} = 0,\\
\frac{dP_x}{d\Omega} &= \left(\frac{dP_{-1}}{d\Omega} - \frac{dP_1}{d\Omega}\right)\frac{1}{\sqrt{2}} \sim \frac{1}{2}(Y_{L-1}+Y_{L1}) \sim\\
&\sim -i\sin\varphi\overline{P_L^1}(\cos\theta),\\
\frac{dP_y}{d\Omega} &= \frac{i}{\sqrt{2}}\left(\frac{dP_1}{d\Omega} + \frac{dP_{-1}}{d\Omega}\right) \sim -\frac{i}{2}(Y_{L1}-Y_{L-1}) \sim\\
&\sim i\cos\varphi\overline{P_L^1}(\cos\theta).
\end{aligned}\right\} \qquad (31.7)$$

$\overline{P}^1_L$ is the normalized associated Legendre function.

We shall assume that we have chosen as the axis by the direction of motion of the colliding particles. The position of the axes x and y is not associated at present with a distinct physical direction. Let us now choose the x-axis so that it lies in the plane of the reaction (this obviously implies that the angle of scattering $\varphi = 0$). In this system of co-ordinates $\frac{dP_z}{d\Omega} = \frac{dP_x}{d\Omega} = 0$, whence follows a very general confirmation that the polarization vector of the particles is always directed perpendicularly to the plane of the reaction. This is also clear from simple geometrical considerations: the polarization vector is obviously a pseudo-vector, and a unit pseudo-vector which can be formed from the unit vectors $\boldsymbol{n}_\alpha$ and $\boldsymbol{n}_{\alpha'}$ is the vector product $[\boldsymbol{n}_\alpha \boldsymbol{n}_{\alpha'}]$, hence it is clear that the polarization vector should be directed along this unit vector. We shall denote this unit vector by $\boldsymbol{k} = [\boldsymbol{n}_\alpha \boldsymbol{n}_{\alpha'}]$, when the general expression for the differential polarization vector can be written in the form*

$$\frac{d\boldsymbol{P}_{\alpha'\alpha}}{d\Omega} = \boldsymbol{k}\frac{\lambda^2}{4}\frac{[2j_1(j_1+1)(2j_1+1)]^{1/2}}{(2j_1+1)(2j_{11}+1)} \times$$
$$\times \sum \mathrm{Re}\left\{i\left[(l_1' s_1' \alpha' \,|\, R^{J_1}_{\pi_1} \,|\, l_1 s \alpha)\right]^* \left[(l_2' s_2' \alpha' \,|\, R^{J_2}_{\pi_1} \,|\, l_2 s \alpha)\right]\right\} \times$$
$$\times (-1)^{j_2 - j_1 - s + J_2 + s_2' + l_2'} (-i)^{-l_1 + l_2 + L} Z(l_1 J_1 l_2 J_2;\ sL) \times$$
$$\times W(j_1 s_1' j_1 s_2';\ j_2 1)\,[(2J_1+1)(2l_1'+1)(2s_1'+1)(2J_2+1)(2l_2'+1) \times$$
$$\times (2s_2'+1)]^{1/2} (l_1' 0 l_2' 0 \,|\, L0)\, X(J_1 l_1' s_1';\ J_2 l_2' s_2';\ LL1)\, \overline{P}^1_L(\cos\theta). \tag{31.8}$$

The summation proceeds with respect to $J_1 J_2 \pi_1 \pi_2 l_1 l_2 l_1' l_2' s_1' s_2'$, s and L

All the quantities occurring in $\frac{dP_{\alpha'\alpha}}{d\Omega}$ are real.

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*Our formula differs from formula (5.2) of [29] by a slight essential factor. The difference is dependent upon the unfortunate normalization of the tensor angular momenta used in [29]. (See [25].) Moreover, there is a difference in the phase factor $i^{l_2' - l_2 + l_1' - l_1}$, associated with an error in [29], an error similar to the one mentioned previously in [19].

Section 32. Fundamental Laws Related to Polarization in Nuclear Reactions

We shall consider the fundamental laws of polarization.

1) As has been indicated, the polarization vector is always directed perpendicular to the plane of the reaction.

2) Polarization is essentially an interference phenomenon.

This is evident from the expression in formula (31.8) which has the form $\sum_{ll'} \mathrm{Re}\left[i f_l^* f_{l'}\right]$, where f_l are the amplitudes of the transitions in the individual channels; the terms with $l = l'$ are equal to zero. Thus, if the specification of a reaction is such that only a single matrix element of the S-matrix is non-vanishing, then polarization is absent.

A number of laws follow directly from the properties of the Racah coefficients and the vector addition coefficients.

3) If in the reaction an S-wave participates (in principle also in the final states), then polarization will be absent.

4) If the reaction goes via the level of a compound nucleus of a definite parity and $J = \frac{1}{2}$ (or $J = 0$, and any parity), then polarization will also be absent.

5) If in the final state the total spin is zero, then the polarization is also zero.

6) In the case of absence of spin-orbital coupling, the polarization is zero.

7) If the specification of the interaction is such that the value of the orbital angular momentum of the initial and final states, or the total angular momentum J, participating in the reaction is limited, then an upper limit can be set for L:

$$L \leqslant 2l_1,\ 2l_1',\ 2J.$$

L should be even if the interfering states have the same parity. Anisotropy of the angular distribution of the reaction obviously excludes the use of rules three and four.

Non-symmetry of angular scattering relative to 90° indicates that there are interference effects, and consequently polarization is possible.

8) Certain predictions can be made concerning the angular distribution of the polarization. For $\theta=0^\circ$ and 180°, $\overline{P}_L^1$ is equal to zero and polarization is absent. If in the reaction, orbital states not higher than $l=1$ participate, then the polarization will be determined by the functions $\overline{P}_1^1(\cos\theta)$ and $\overline{P}_2^1(\cos\theta)$, i.e. it should be expected that it will be a maximum for angles θ within the range 45° and 135°.

The formula given for polarization is very unwieldy. However, this disadvantage is associated with its extreme generality. If particles with low spins participate in the reaction, and only a small number of values for the orbital angular momentum occur, then the formula can be simplified and in every particular case the coefficients can be established in the form of numerical factors by means of the tables. It is particularly easy to use this formulae for angular distributions and polarizations, if the numerical tables given in Appendix II for the coefficients W, Z and X are used.

We shall give as an example of application of formula (31.8) an analysis of the reaction $Li^6(n\alpha)H^3$. Interpretation [31] of the data for this reaction indicates that in the energy region about 270 keV, its angular distribution and energy dependence can be explained, if it is assumed that the following elements of the S-matrix are non-zero: the left hand portion of the equation in our notation and the right hand side in the notation of [31]:

$$\left(1\,\tfrac{1}{2}\middle|S_-^{\frac{3}{2}}\middle|1\,\tfrac{1}{2}\right)=\frac{\tau}{\sqrt{2}},\quad \left(2\,\tfrac{1}{2}\middle|S_+^{\frac{3}{2}}\middle|0\,\tfrac{3}{2}\right)=a,$$

$$\left(0\,\tfrac{1}{2}\middle|S_+^{\frac{1}{2}}\middle|0\,\tfrac{1}{2}\right)=b.$$

On the basis of this data and the general principles formulated above, certain predictions can be made concerning the polarization of the tritium nuclei arising as a result of the reaction. Since the states with different spins in the original states do not interfere (see formula (31.8)), then the polarization can be specified by the interference states τ and b, and the sum with respect to L in (31.8) for this case amounts to a single term with $L=1$. Using the properties of the W and X coefficients and the tables, it is not difficult to find that:

$$W\left(\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2};\,01\right)=\tfrac{1}{2},\quad W\left(1\,\tfrac{3}{2}\,0\,\tfrac{1}{2};\,\tfrac{1}{2}\,1\right)=\frac{1}{\sqrt{6}};$$

$$X=\left(\tfrac{3}{2}\,1\,\tfrac{1}{2};\,\tfrac{1}{2}\,0\,\tfrac{1}{2};\,111\right)=\frac{1}{\sqrt{216}},$$

and formula (31.8) gives

$$\frac{dP}{d\Omega}=k\lambda^2(864)^{-\frac{1}{2}}\left[i(\tau b^*-\tau^*b)\right]\overline{P}_1^1(\cos\theta)=k\lambda^2\frac{2}{\sqrt{864}}\,|\tau^*b|\sin\eta\,\overline{P}_1^1(\cos\theta).$$

The angular distribution does not allow the determination of the phase shift η, upon which the magnitude of the polarization depends. On the basis of data from angular distributions, however, it is easy to estimate the maximum value of the polarization which can be expected for this reaction.

From the example discussed, it is obvious that the general principles formulated permit a sufficiently definite opinion to be given concerning the angular dependence of polarization and its magnitude, i.e. they permit a selection to be made of the most advantageous conditions for measurement of polarization. The measurement of polarization makes it possible to obtain information about the phase shifts of the elements of the S-matrix.

CHAPTER X

REACTIONS INVOLVING PHOTONS

Section 33. General Formulae

The mathematical treatment given also applies when photons* participate in a reaction. In order to understand those special features which arise in this case, it is necessary to consider the quantum mechanics of the photon. We will only consider this briefly, however, since only certain results of this theory are necessary for our narrow practical purposes. The student, desirous of studying the problem in more detail, should for example read the first chapter of the monograph [20], where the quantum mechanics of the photon is stated simply.

Henceforth we shall use the following results of quantum mechanics of the photon:

1) A photon, like any other particle, has a wave function f. It is related very simply to vector potential and is a vector quantity. The latter shows that the spin of the photon is equal to unity. f_z is the eigen function of the operator of the projection of the photon spin with an eigen-value $\mu=0$; the combinations $-\frac{1}{\sqrt{2}}(f_x+if_y)$ and $\frac{1}{\sqrt{2}}(f_x-if_y)$ are the eigen functions of the operator of the projection of the photon spin with eigen-values of + 1 and - 1 respectively.

2) The wave function of a photon satisfies the condition: $(n.\,f(n))=0$, where n is a unit vector in the direction of propagation of the photon.

*See [22, 18, 26].

From this latter result emerge the special characteristics of the general formulae describing reactions with photons.

Our problem is to express this condition in such a form that it should permit us, by a simple modification of the general formulae obtained for particles with a non-zero rest mass, to obtain similar formulae for photons.

For particles with a spin of 1, the wave function with a finite orbital angular momentum l, its projection m and spin projection μ, has the form:

$$(\theta\varphi\mu' \mid lm\mu) = Y_{lm}(\theta,\ \varphi)\,\delta_{\mu'\mu}. \qquad (33.1)$$

where θ and φ are the angles defining the direction of the momentum of the particle. The wave function of a particle in the same representation, but with a finite total angular momentum g and projection m_g is obtained from function (33.1) by the rule of vector addition of angular momenta

$$(\theta\varphi\mu' \mid lgm_g) = \sum_{m\mu}(\theta\varphi\mu' \mid lm\mu)\,(lm1\mu \mid gm_g). \qquad (33.2)$$

For a given g and m_g, there are three linear independent functions with $l=g$ and $l=g\pm1$. The eigen function of the total angular momentum g, in the general case, is any linear combination of these three functions.

In the case of a photon, in contrast to the normal particle with a spin of 1, the wave functions should also satisfy the condition $n.\,f(n)=0$. Therefore, in order to obtain the wave function of a photon describing the state with a finite total angular momentum and its projection, it is necessary to construct the linear combinations of function (33.2) satisfying this additional condition. Hence it follows, that for a photon there will be not three different states with the fixed quantum numbers g and m_g, but only two.

If we find this combination

$$\sum_l C^l(p)\,(\theta\varphi\mu' \mid lgm_g) = (\theta\varphi\mu' \mid pgm_g)$$

(the index p can assume two values), then the coefficient $C^l(p)$ can be considered as the transformation $(l \mid p)$, converting the formulae valid for the normal particles, into formulae valid for photons. Thus, our problem is that of determining $(l \mid p)$.

Let us find $C^l(p)=(l|p)$ from the following condition:

1) The functions $(\theta\varphi\mu'|pgm_g)$ should satisfy the condition $\boldsymbol{n}, \boldsymbol{f}(\boldsymbol{n})=0$. We shall formulate this condition in a representation for which we shall use

$$(\boldsymbol{n}, \boldsymbol{f}(\boldsymbol{n}))=\cos\theta f_z(\boldsymbol{n})+\cos\varphi\sin\theta f_x(\boldsymbol{n})+\sin\varphi\sin\theta f_y(\boldsymbol{n})=$$
$$=\cos\theta f_z(\boldsymbol{n})+\frac{1}{2}(f_x+if_y)\sin\theta e^{i\varphi}+\frac{1}{2}(f_x-if_y)\sin\theta e^{i\varphi}=$$
$$=\sqrt{\frac{4\pi}{3}}(Y_{10}f_0+Y_{11}f_1+Y_{1-1}f_{-1})=0,$$

where f_μ is the eigen function of the operator of the projection of spin in the eigen representation. Thus, this condition can be written in the form:

$$\sum_\mu Y_{1\mu'}(\theta, \varphi)(\theta\varphi\mu'|pgm_g)=0. \tag{33.3}$$

2) The functions $(\theta\varphi\mu'|pgm_g)$ should be eigen functions of the parity operator.

3) The functions $(\theta\varphi\mu'|pgm_g)$ should be orthogonal and normalized.

We observe that the functions $(\theta\varphi\mu'|pgm_g)$, in accordance with the properties listed, describe an electromagnetic wave with a finite total angular momentum and parity. Such states in the classical theory of the electromagnetic field are called multipoles.

The functions $(\theta\varphi\mu'|lgm_g)$ are eigen-functions of the parity operator:

$$P(\theta\varphi\mu'|lgm_g)=(-1)^{l+1}(\theta\varphi\mu'|lgm_g). \tag{33.4}$$

The factor $(-1)^l$ is obtained on account of the transformation $\theta\to\pi-\theta, \varphi\to\varphi+\pi$ in the orbital part of the function, and (-1) on account of the spin part of the wave function. Since $l=g\pm 1, g$, then the state $l=g$ pertains to one parity, and $l=g\pm 1$ to the other. There the functions can be either $(\theta\varphi\mu'|ggm_g)$ or $(\theta\varphi\mu'|g\pm 1gm_g)$. Hence we find one of the functions required:

$$(\theta\varphi\mu'|ggm_g)=(\theta\varphi\mu'|0gm_g). \tag{33.5}$$

In (33.5) we have followed the notation used in the literature and made the index p equal to zero. Such states are called magnetic multipoles; their parity is equal to $(-1)^{g+1}$. Actually, for $g=1$, one talks about the magnetic dipole state, for $g=2$ about the magnetic quadrupole state and so forth.

Function (33.5), in accordance with its definition from (33.2), satisfies our third condition. It is not difficult to verify that it also satisfies the first condition.

$$\sum_\mu Y_{1\mu'}(\theta, \varphi)\sum_{m\mu} Y_{gm}(\theta, \varphi)\delta_{\mu'\mu}(gm1\mu|gm_g)=$$
$$=\sum_\mu Y_{gm_g-\mu}Y_{1\mu}(gm_g-\mu 1\mu|gm_g).$$

Replacing the product of the sphericcal harmonic functions by means of formula (30.5), we find

$$\sum_\mu\sum_{L=l-1}^{l+1}\sum_{M=-L}^{L}\left[\frac{(2l+1)(2l+1)}{4\pi(2L+1)}\right]^{\frac{1}{2}}(10g0|L0)\times$$
$$\times(LM|1\mu gm_g-\mu)(1\mu gm_g-\mu|gm_g)Y_{LM}.$$

The two latter vector addition coefficients, by summation with respect to μ give the product $\delta_{Lg}\delta_{Mm_g}$, and since the coefficient $(10g0|g0)$ is equal to zero, then the second condition will be fulfilled.

Let us find the second function, having a parity $(-1)^g$ (in the literature the index $p=1$) is used and these states are called electric multipoles:

$$(\theta\varphi\mu'|1gm_g)=C^{g-1}(1)(\theta\varphi\mu'|g-1gm_g)+$$
$$+C^{g+1}(1)(\theta\varphi\mu'|g+1gm_g).$$

Using the condition (33.3) and making similar calculations, we find

$$C^{g-1}(1)=\sqrt{\frac{g+1}{g}}C^{g+1}(1),$$

and by summation over all initial states we obtain the final expression for the wave function of an electric multipole:

$$(\theta\varphi\mu'|1gm_g)=\sqrt{\frac{g}{2g+1}}(\theta\varphi\mu'|g-1gm_g)+$$
$$+\sqrt{\frac{g+1}{2g+1}}(\theta\varphi\mu'|g+1gm_g).$$

The coefficients $C^l(p)$ can be written in the general form:

$$C^l(p) = (l \mid p) = -\sqrt{2}\,(g-111 \mid l0)\,\delta(l,\ p)\,(-1)^p. \qquad (33.6)$$

where the symbol $\delta(l,\ p)$ has the following values:

$$\delta(l,\ p) = \begin{cases} 1 & \text{for } l=g \\ 0 & \text{for } l \neq g \end{cases} \quad \text{for } p=0,$$
$$\delta(l,\ p) = \begin{cases} 0 & \text{for } l=g \\ 1 & \text{for } l \neq g \end{cases} \quad \text{for } p=1. \qquad (33.7)$$

Transformation (33.6) is of the greatest value since it can be used to transform many formulae valid for reactions with particles into formulae valid for reactions involving photons.

By means of (33.6) a transformation is made from the variable l to a variable characterizing a type of electromagnetic radiation (electric, magnetic), where (and it is necessary to emphasize this) the states of a photon are given by the total angular momentum (orbital angular momentum plus spin), whereas the states of particles have been given by the total spin. This shows, that in order to obtain formulae valid for reactions with particles, it is necessary to carry out a further transformation, changing the order of addition of the angular momenta:

$$(\hat{j}_{\rm I} + \hat{j}_{\rm II} = \hat{s}) + \hat{l} \rightarrow \hat{j}_{\rm I} + (\hat{j}_{\rm II} + \hat{l} = \hat{g}) = \hat{J}$$

And so forth.

But in accordance with the the result in Section 29, this transformation is:

$$\sqrt{(2g+1)(2s+1)}\,W(l1Jj_{\rm II} \mid gs) = (s \mid g).$$

Here we assume $j_{\rm I}=1$, i.e. the case is considered when the photon is particle I.

We shall now derive the final formula for transformation from formulae for reactions with particles characterized by variables of spin of the channel s, to formulae describing reactions with photons. The states of the photons are given in the form of electric and magnetic multipoles.

1) It is necessary to transform the amplitude of the process

$$(l's'\alpha' \mid R^J_\pi \mid ls\alpha) \rightarrow \sum_{gp} (l's'\alpha' \mid R^J_\pi \mid pg\alpha)(pg \mid sl).$$

$$(sl \mid gp) = (s \mid g)(l \mid p) =$$
$$= -\sqrt{2}\,(-1)^p \sqrt{(2g+1)(2s+1)}\,W(l1Jj_{\rm II};\ gs) \times$$
$$\times (g-111 \mid l0)\,\delta(l,\ p),$$

the meaning of the symbols was given previously.

2) For summation over the initial states, it is necessary to take into account the fact that a photon with a fixed g does not have 3 states, but only 2. Thus, we obtain from formula (30.6) a general formula for angular distributions in nuclear reactions involving photons [26]*.

a) Photoreactions (photons incident, particles emitted)

$$\frac{d\sigma}{d\Omega} = \frac{\lambda^2}{2(2j_{\rm II}+1)} \sum \frac{(-1)^{s'-j_{\rm II}-1}}{4} \times$$
$$\times \operatorname{Re}\left\{(l_1's'\alpha' \mid R^{J_1}_{\pi_1} \mid p_1g_1\alpha)^* (l_2's'\alpha' \mid R^{J_2}_{\pi_2} \mid p_2g_2\alpha)\right\} \times$$
$$\times (-1)^{p_1+p_2} (i)^{-l_2'+l_1'-L}\, Z(l_1'J_1l_2'J_2;\ s'L) \times$$
$$\times Z\ (g_1J_1g_2J_2;\ j_{\rm II}L)\,P_L(\cos\theta). \qquad (33.8)$$

Here the summation is over $J_1J_2g_1g_2p_1p_2l_1'l_2's'$ and L.

b) Emission of photons (particles incident, photons emitted)

$$\frac{d\sigma}{d\Omega} = \frac{\lambda^2}{(2j_{\rm I}+1)(2j_{\rm II}+1)} \times$$
$$\times \sum \frac{(-1)^{j_2-s+1}}{4} \operatorname{Re}\left\{(p_1'g_1'\alpha' \mid R^{J_1}_{\pi_1} \mid ls\alpha)^* (p_2'g_2'\alpha' \mid R^{J_2}_{\pi_2} \mid l_2s_2\alpha)\right\} \times$$
$$\times (-1)^{p_1'+p_2'} Z_\gamma (g_1'J_1g_2'J_2;\ j_2L)\,(i)^{-l_2+l_1-L} \times$$
$$\times Z(l_1J_1l_2J_2;\ sL)\,P_L(\cos\theta). \qquad (33.9)$$

*Our formulae differ from the formulae in the paper [26] by a phase factor chosen from the condition of invariancy of the S-matrix under time reversal (see Section 21).

Here the summation is over $J_1 J_2 g_1 g_2 p_1 p_2 l_1' l_2' s'$ and L.

c) Scattering of photons

$$\frac{d\sigma}{d\Omega}=\frac{\lambda^2}{2(2j_{II}+1)}\times$$
$$\times\sum\frac{(-1)^{j_2-j_{II}}}{4}\operatorname{Re}\{(p_1'g_1'\alpha'|R^{J_1}_{\pi_1}|p_1g\alpha)^*(p_2'g_2'\alpha'|R^{J_2}_{\pi_2}|p_2g_2\alpha)\times$$
$$\times(-1)^{p_1'+p_2'+p_1+p_2}Z_\gamma(g_1'J_1g_2'J_2;j_{II}L)Z_\gamma(g_1J_1g_2J_2;j_2L)P_L(\cos\theta)\}. \tag{33.10}$$

Here the summation is over $J_1 J_2 g_1 g_2 g_1' g_2' p_1' p_2' p_1 p_2$ and L.

It is not difficult to obtain an expression for polarization of particles arising as a result of photoreactions. Transformation of formula (31.8) gives

$$\frac{d\boldsymbol{P}}{d\Omega}=\boldsymbol{k}\frac{\lambda^2}{4}\frac{[2j_1(j_1+1)(2j_1+1)]^{1/2}}{2(2j_{II}+1)}\times$$
$$\times\sum\operatorname{Re}\{i(l_1's_1'\alpha'|R^{J_1}_{\pi_1}|p_1g_1\alpha)^*(l_2's_2'\alpha'|R^{J_2}_{\pi_2}|p_2g_2\alpha)\}\times$$
$$\times(-1)^{j_2-j_1-j_{II}+J_2+s_2'+l_2'-1}Z_\gamma(g_1J_1g_2J_2;j_{II}L)(-1)^{p_1+p_2}\times$$
$$\times W(j_1s_1'j_1s_2';j_21)\times$$
$$\times[(2J_1+1)(2l_1'+1)(2s_1'+1)(2J_2+1)(2l_2'+1)(2s_2'+1)]^{1/2}\times$$
$$\times(l_1'0l_2'0|L0)X(J_1l_1's_1';J_2l_2's_2';LL1)\overline{P}^1_L(\cos\theta). \tag{33.11}$$

The summation is over $J_1, J_2, g_1, g_2, p_1, p_2, l_1', l_2', s_1', s_2'$ and L.

The properties of expressions (33.8), (33.9), (33.10) and (33.11), are similar to the properties discussed above for the corresponding expressions for collision of particles.

In the summations of (33.8)-(33.10), only those terms for which $p_1+p_2+g_1+g_2-L$ is an even number are non-zero.

As an exercise, we suggest that the student, by applying the formulae given here and the tables in Appendix II, obtain:

1. The angular distribution in the photoproduction of π-mesons at nucleons, assuming that the amplitudes E_{11} for photoproduction of S-state mesons as a result of absorption of a photon in the electric dipole state $(g=1,\ p=1,\ J=1/2)$, and that the amplitudes M_{11} and M_{13} for photoproduction of P-state mesons as a result of absorption of photons in the magnetic dipole state, are respectively, $(g=1,\ p=0,\ J=1/2)$ and $(g=1,\ p=0,\ J=3/2)$.

Answer:

$$\frac{d\sigma}{d\Omega}=\frac{1}{8k^2}\{[|E_{11}|^2+|M_{11}|^2+2|M_{13}|^2]P_0(\cos\theta)+$$
$$+\operatorname{Re}[-2E_{11}^*M_{11}+2E_{11}^*M_{13}]P_1(\cos\theta)+$$
$$+[-|M_{13}|^2-2\operatorname{Re}(M_{11}^*M_{13})]P_2(\cos\theta)\}.$$

2. The angular distribution for the Compton effect by a nucleon, assuming that the photons are scattered only in the electric and magnetic dipole states.

Answer:

$$\frac{d\sigma}{d\Omega}=\frac{1}{32k_\gamma^2}\{[4|E_{1/2}|^2+7|E_{3/2}|^2+4|M_{1/2}|^2+7|M_{3/2}|^2-$$
$$-2\operatorname{Re}(E_{1/2}^*E_{3/2}+M_{1/2}^*M_{3/2})]+$$
$$+4\operatorname{Re}[E_{1/2}^*(2M_{1/2}+M_{3/2})+E_{3/2}^*(M_{1/2}+5M_{3/2})]\cos\theta+$$
$$+3[|E_{3/2}|^2+|M_{3/2}|^2+2\operatorname{Re}(E_{1/2}^*E_{3/2}+M_{1/2}^*M_{3/2})]\cos^2\theta\},$$

where $E_{1/2}$ and $E_{3/2}$ are the amplitudes of the electric dipole scattering in the states with a total angular momentum of 1/2 and 3/2 respectively, and $M_{1/2}$ and $M_{3/2}$ are the same for magnetic dipole scattering.

Section 34. Relationship between Photoproduction Processes, Scattering of π-mesons and the Compton Effect by a Nucleon

In Section 26 the determination of the parameters of the S-matrix was briefly discussed, and it was indicated that the treatment given there was of a very general nature. It is, in fact, also used for reactions in which a change of nature of the particles takes place.

In this paragraph we shall determine the parameters of the S-matrix describing the fundamental process of π - meson physics, and we shall establish the relationship between these processes which result only from the general properties of the S-matrix given in Chapter VI.

Let us consider the series of reactions:

$\gamma+N\to\gamma'+N'$ — Compton effect by a nucleon.

$\gamma+N\to\pi+N'$ — Photoproduction of mesons.

$\pi+N\to\gamma+N'$ — Radiative capture.

$\pi+N\to\pi'+N'$ — Scattering of mesons.

Here the symbols N, N' designate nucleons (either protons or neutrons), and π-mesons (positive, negative, neutral), γ-photons. All the initial and final states we shall consider as different states of a single quantum-mechanical system.

If the energy of the photons is limited to below 300 MeV, then the transitions between the states (γN), $(\gamma' N')$, $(N\pi)$, $(N'\pi')$ practically exhaust the possible processes. Calculation of the remaining channels is slightly influenced by the results discussed below. The channel with formation of electron pairs requires special consideration. The S-matrix element describing this transition is by no means so small, that in accordance with Section 24 it causes no diffraction scattering of the photons by the nucleons. It can be shown, however [35], that if angles of scatter of the photons by the nucleons less than $\sim\frac{m_e c^2}{E_\gamma}$, are not considered, then this

effect can be neglected.

Let us write the S-matrix of the processes we have listed:

$$(k'\alpha'|S^{J\pi}|k\alpha)\delta_{J'J}\delta_{M'M}\delta_{\pi'\pi}\delta_{T_3'T_3}.$$

Here J is the total angular momentus of the system, M its projection, π the parity of the state, T_3 the projection of the total isotopic spin of the system, k all the remaining quantum numbers characterizing the channel α; α assumes two values, corresponding to (γN) and (πN).

In the case when α corresponds to the channel (πN), the index k takes into account the following quantum numbers: the orbital angular momentus l, and isotopic spin of the system. For a given parity π and angular momentum J, l is single-valued, since $l=J\pm\frac{1}{2}$ and $\pi=(-1)^{l+1}$. The isotopic spin of the system can assume the two values $T=\frac{1}{2}$ and $T=\frac{3}{2}$. In the case when α corresponds to the channel (γN), index k takes into account the quantum numbers g and p, characterizing the multipolarity of the photon. And since $\pi=(-1)^{g+p+1}$, and at the same time $l=J\pm\frac{1}{2}$, then for given J and π, only two states of the photon are possible.

Thus, $(k'\alpha'|S^{J\pi}|k\alpha)$ is represented by a four-series square matrix. We shall write it in the form $S_{\mu\nu}$, where the indices μ and ν take all four values corresponding to the following states:

μ or $\nu=1$ - nucleon + photon in a state of the magnetic type with $g=l$ (the parity of the state is designated by the quantum number l of the orbital angular momentus of the meson-nucleon system).

μ or $\nu=2$ - nucleon + photon in a state of the electric type with $g=l+1$ $(p=1)$.

μ or $\nu = 3$ - nucleon + meson in a state with isotopic spin equal to 1/2.

μ or $\nu = 4$ - nucleon + meson in a state with isotopic spin equal to 3/2.

Without further use of the properties of the S-matrix, we have, for a given total angular momentum and parity, 32 effective parameters by which the S-matrix is expressed.

Let us use the unitarity and symmetry of the S-matrix:

$$\sum_{\nu} S^{*}_{\mu\nu} S_{\nu\mu'} = \delta_{\mu\mu'}, \qquad S_{\mu\nu} = S_{\nu\mu}. \tag{34.1}$$

The symmetry follows from the property of time reversal and from the circumstance that amongst the quantum numbers k and α, there is none which should change sign as a result of substitution of $t \to -t$. Representing $S_{\mu\nu}$ in the form $r_{\mu\nu} e^{i\varphi_{\mu\nu}}$ and substituting this expression in relationship (34.1), we find the system of transcendent equations connecting the parameters $r_{\mu\nu}$ and $\varphi_{\mu\nu}$ which we have introduced. This system can be solved by the method of successive approximations, using the simplest assumption, actually as the result of experiment, that the matrix elements are related to one another thus:

$$(S_{11}-1 \sim S_{12} \sim S_{22}-1): (S_{13} \sim S_{14} \sim S_{23} \sim S_{24} \sim S_{34}) : (S_{33} \sim S_{44}) = \frac{e^2}{\hbar c} : \sqrt{\frac{e^2}{\hbar c}} : 1, \tag{34.2}$$

where $\frac{e^2}{\hbar c} = \frac{1}{137}$, is the fine structure constant. Relationship (34.2) is obviously unbalanced absolutely in the vicinity of the threshold of photoproduction. We shall not discuss this small region here. Below the threshold only the elements S_{11}, S_{12}, S_{21} and S_{22} are non-zero, and the parametrization does not differ from that given in Section 26. Consequently, we shall consider the region of energy above the threshold for photoproduction of π-mesons. As a first

approximation we shall keep in each of the equations of (34.1), only the terms involving one power $\sqrt{\frac{e^2}{\hbar c}}$. For μ and μ', equal to 3 or 4, we find

$$S^{*}_{33} S_{33} = 1 \quad \text{и} \quad S^{*}_{44} S_{44} = 1,$$

hence

$$S_{33} = e^{2i\gamma_3}, \qquad S_{44} = e^{2i\gamma_4}. \tag{34.3}$$

The effective parameters γ_3 and γ_4 introduced here are not the same as the phase shifts for the scattering of π-mesons by nucleons in a state with finite isotopic spin, total angular momentum and parity.

When one of the indices μ or μ' is equal to 3 or 4, and the other is equal to 1 or 2, we find

$$\left.\begin{aligned} S_{13} + S^{*}_{13} S_{33} = 0, \quad S_{14} + S^{*}_{14} S_{44} = 0, \\ S_{23} + S^{*}_{23} S_{33} = 0, \quad S_{24} + S^{*}_{24} S_{44} = 0. \end{aligned}\right\} \tag{34.4}$$

From (34.4) and (34.3) we obtain

$$\left.\begin{aligned} S_{13} = i r_{13} e^{i\gamma_3}, \quad S_{14} = i r_{14} e^{i\gamma_4}, \\ S_{23} = i r_{23} e^{i\gamma_3}, \quad S_{24} = i r_{24} e^{i\gamma_4}, \end{aligned}\right\} \tag{34.5}$$

where $r_{\mu\nu}$ are effective parameters.

Relationship (34.5) expresses an important connexion between the processes of photoproduction and scattering of π-mesons [9]. They considerably simplify analysis of experiments on meson photoproduction. For example, by applying the formulae of the preceding paragraph, a number of conclusions can be made concerning interference terms, knowing the energy relationship for the phase shifts involved in meson scattering. And, alternatively, by studying meson photoproduction, it is possible to verify the accuracy of the results of phase shift analysis of scattering experiments.

When the indices μ and μ' are equal to 1 or 2, then equation (34.1), in the first approximation, can be written in the following manner:

$$\left.\begin{aligned} S_{11}^{*} - 1 + S_{11} - 1 &= -S_{13}^{*}S_{31} - S_{14}^{*}S_{41}, \\ S_{12}^{*} + S_{12} &= -S_{13}^{*}S_{32} - S_{14}^{*}S_{42}, \\ S_{22}^{*} - 1 + S_{22} - 1 &= -S_{23}^{*}S_{32} - S_{24}^{*}S_{42}. \end{aligned}\right\} \qquad (34.6)$$

Let us deduce the scattering amplitude: $iR_{\mu\nu} = S_{\mu\nu} - \delta_{\mu\nu}$, and let us use relationship (34.5); then relationship (34.6) can be written briefly as

$$\operatorname{Im} R_{ab} = \frac{1}{2}(r_{a3}r_{3b} + r_{a4}r_{4b}). \qquad (34.7)$$

where a and b assume the values 1 and 2, i.e. the imaginary part of the amplitude for the Compton effect is expressed by the modulus of the amplitude for photoproduction of π - mesons.

Relationship (34.7) substantially facilitates analysis of experiments on scattering of photons by nucleons. Using the general property of the W, Z - coefficients and vector addition coefficients, it is not difficult to show that substitution of the indices $1 \rightleftarrows 2$ under the summation sign in formula (33.10) does not alter the value of $\frac{d\sigma}{d\Omega}$. Hence, it is possible to replace the symbol Re by the product $(p_1'g_1'\alpha' | R_{\pi_1}^{J_1} | p_1 g_1 \alpha)^* (p_2'g_2'\alpha' | R_{\pi_2}^{J_2} | p_2 g_2 \alpha)$ If now $R_{\mu\nu}$ is written in the form $R_{\mu\nu}^{I} + iR_{\mu\nu}^{II}$, where $R_{\mu\nu}^{I}$ and $R_{\mu\nu}^{II}$ are real values, then $\frac{d\sigma}{d\Omega}$ is divided into two components; in the one only the elements $R_{\mu\nu}^{I}$ occur, and in the other only $R_{\mu\nu}^{II}$. In accordance with (34.7) this shows that the first component is expressed only through a parameter subject to determination in experiments on the Compton effect, and the second one only through the modulus of the amplitude for photoproduction of mesons. From this result a number of the properties of the differential cross-section of the Compton effect can be obtained without the use of the detailed theory of the phenomenon.

APPENDICES

APPENDIX I

(To Part One)

In Chapter II methods of calculation and graphical construction of kinematic relationships characterizing nuclear interactions involving two secondary particles were given. The number of such interactions, even if limited only to transformations of elementary particles, is extremely large, and moreover these interactions are investigated over extremely wide ranges of energy of the bombarding particles. Consequently, we are not able to present here, concrete data for all such interactions, and we are including in the Appendix only a few fairly typical examples.

As examples of nuclear reactions at low energies, we shall present a few reactions used for the production of monoenergetic neutrons, such as *dn* -reactions:

1) $D(dn)He^3$, 2) $T(dn)He^4$, 3) $C^{12}(dn)N^{13}$

and *pn* -reactions

1) $T(pn)He^3$ and 2) $Li^7(pn)Be^7$.

In Figure 1* are shown the maximum and minimum energies of neutrons (corresponding to their emmission through angles of 0 and 180°) for different energies of the bombarding particles in the specified reactions. In Figures 2 and 3, certain characteristics are shown of the reaction $T(dn)He^4$ for energies of bombarding deuterons of $W_d = 0.5 - 3$ MeV.

In Figure 2, the relationship between the angle of emission of α-particles and the angle of emission of the neutron: $\vartheta_\alpha = f(\vartheta_n)$ is given, and also the relationship between the

*Figures 1-8 in the Appendix are taken from a paper by Hansen, Taschek and Williams, Rev.Mod.Phys., 21, 635 (1949).

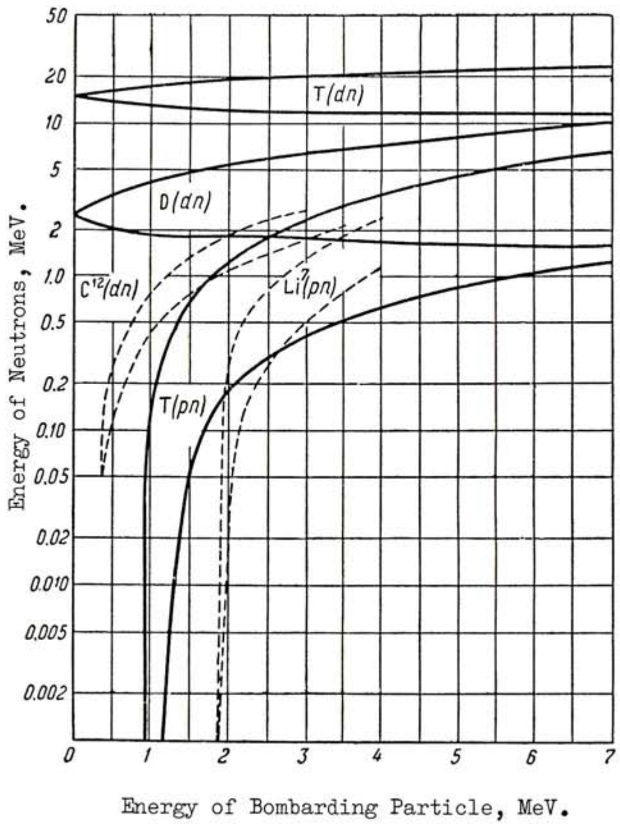
50
20
10
5
2
1.0
0.5
0.2
0.10
0.05
0.02
0.010
0.005
0.002
T(dn)
D(dn)
C[12](dn)
Li[7](pn)
T(pn)
0
1
2
3
4
5
6
7
Energy of Neutrons, MeV.
Energy of Bombarding Particle, MeV.

Fig. 1

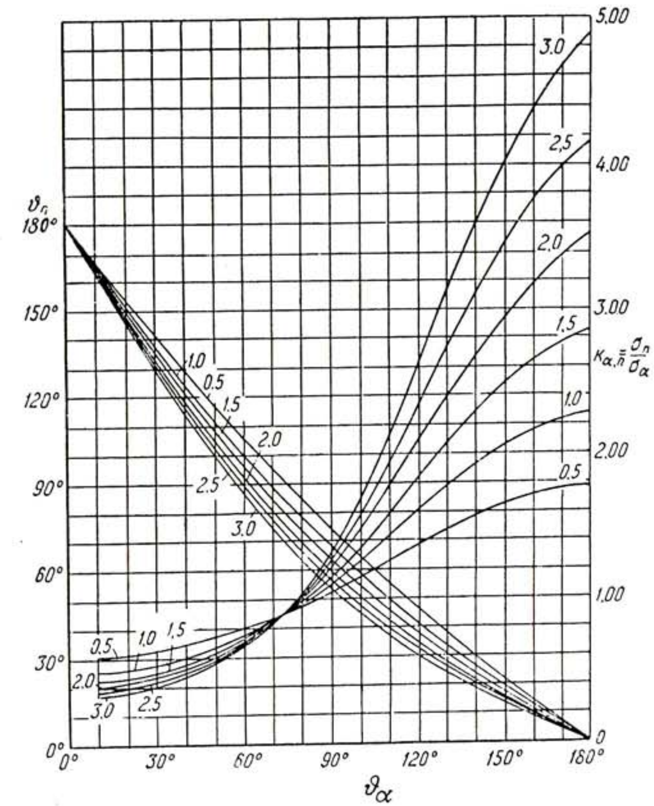
5.00
4.00
3.00
2.00
1.00
0
180°
150°
120°
90°
60°
30°
0°
0°
30°
60°
90°
120°
150°
180°
3.0
2.5
2.0
1.5
1.0
0.5

Fig. 2

differential cross-sections of the emitted neutrons and the α -particles for various ϑ_α :

$$k_{\alpha n} = \frac{\sigma_n(\vartheta_n)}{\sigma_\alpha(\vartheta_\alpha)} = f(\vartheta_\alpha).$$

Figure 3 shows the relationship between the kinetic energy of the neutrons and their angle of emission: $W_n = f(\vartheta_n)$. The numbers by the curves in these figures correspond to the energies of the bombarding neutrons.

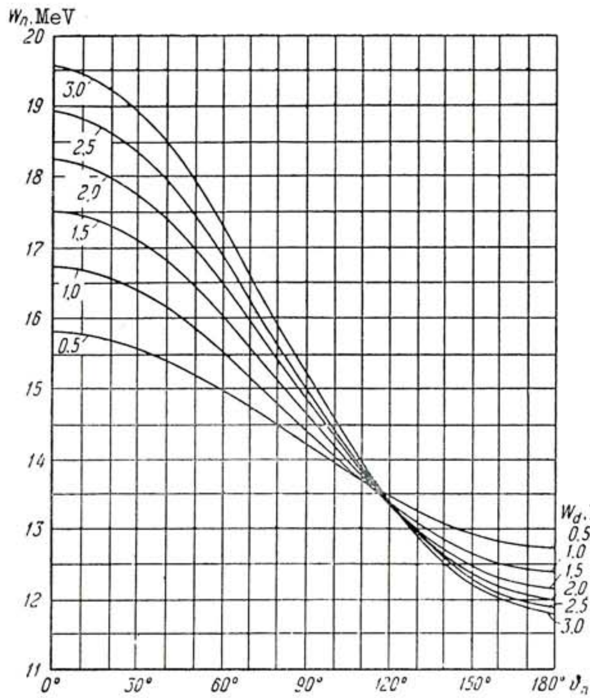

Fig. 3

The inter-relationship between the energies of the neutrons emitted in the reactions listed above and their angles of emission in the L- and CM-systems, and the energy of the bombarding particles, can be defined by means of the nomograms plotted in Figures 4-8. In these nomograms, the

continuous concentric semicircles with a radius $\sqrt{W_n}$ give a number of values for the energies of the neutrons W_n in the L-system. The angles ϑ_n in the L-system are shown by continuous radial lines, traversing the energy semicircles. The centres of the dotted (non-concentric) semicircles with the same radius $\sqrt{W_n}$ are displaced tc the right by an amount $\sqrt{\frac{m_n}{2}}\,V$ from the centre of the continuous semicircle, and intersect the horizontal axis for different values of energy of the bombarding particles - W_d or W_p. The angular co-ordinates of points on the dotted semicircles give the values of the angles $\tilde{\vartheta}_n$ in the CM-system, and the dotted radial lines give the geometrical position of points with a given value of $\tilde{\vartheta}_n$.

Let us give an example of the use of the nomogram. Suppose we require to find the energy of the neutron W_n emitted in the reaction D (dn) He ^{3} at an angle $\vartheta_n = 60°$, if the energy of the bombarding deuteron $W_d = 2$ MeV. We find in Figure 4 on the horizontal scale W_d the value 2 MeV. and

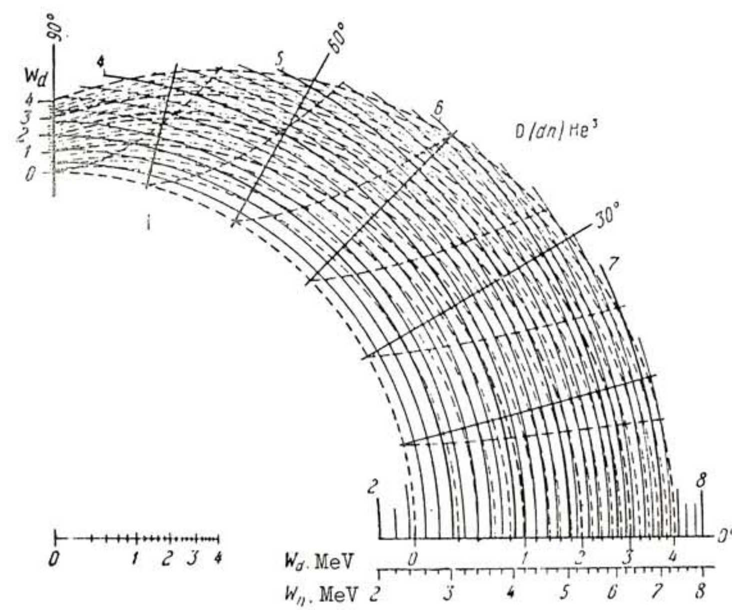

Fig. 4

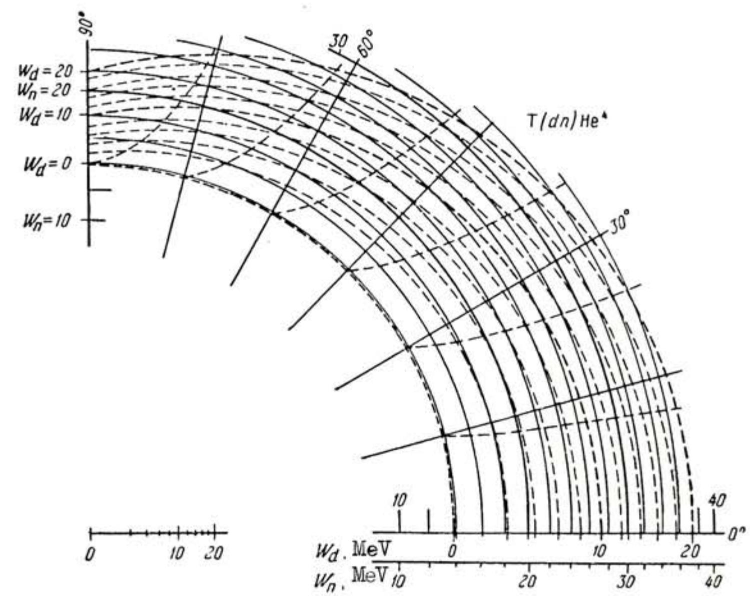

Fig. 5

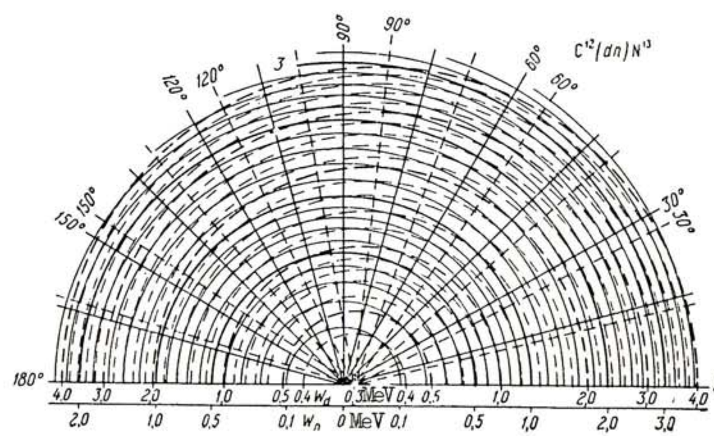

Fig. 6

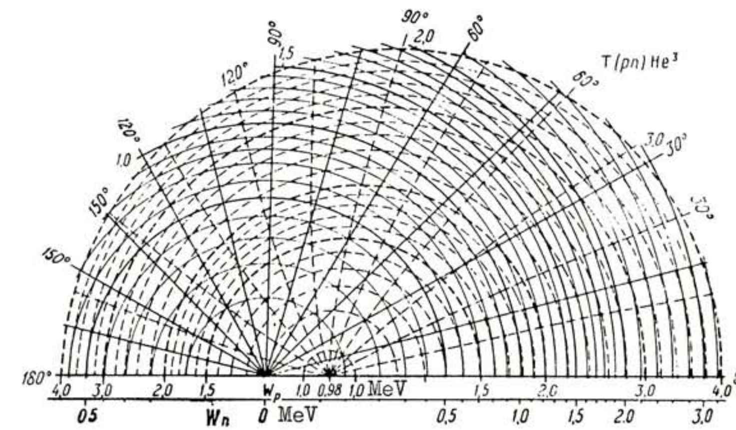

Fig. 7

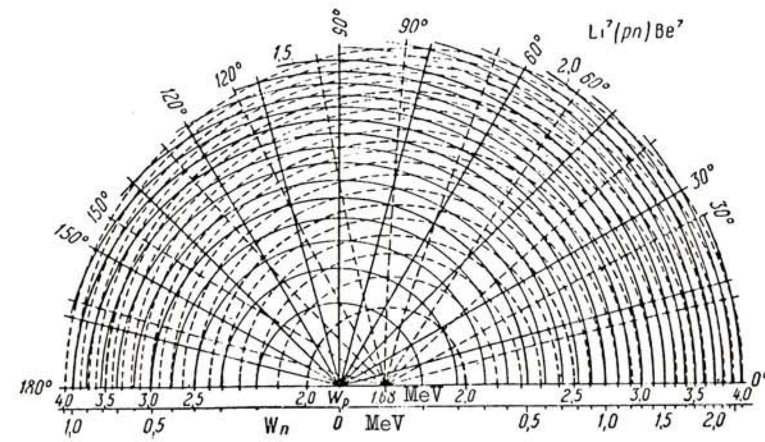

Fig. 8

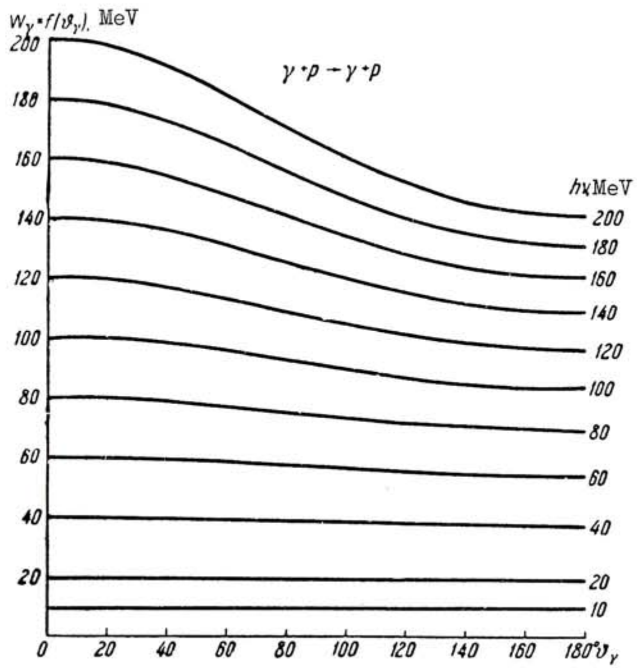
$W_\gamma = f(\vartheta_\gamma)$, MeV
$\gamma + p \to \gamma + p$
$h\nu$, MeV
200
180
160
140
120
100
80
60
40
20
10
0
20
40
60
80
100
120
140
160
180° ϑ_γ

Fig. 9

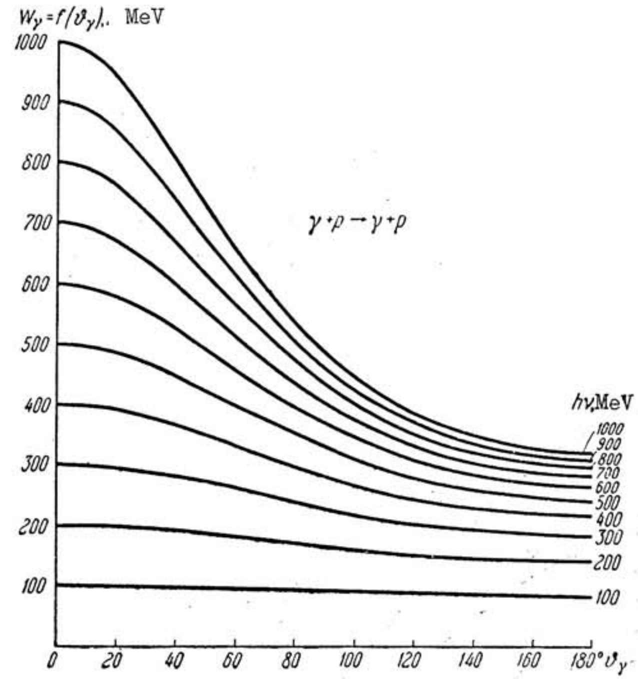
$W_\gamma = f(\vartheta_\gamma)$, MeV
1000
900
800
700
600
500
400
300
200
100
$\gamma + p \to \gamma + p$
$h\nu$, MeV
0
20
40
60
80
100
120
140
160
180° ϑ_γ

Fig. 10

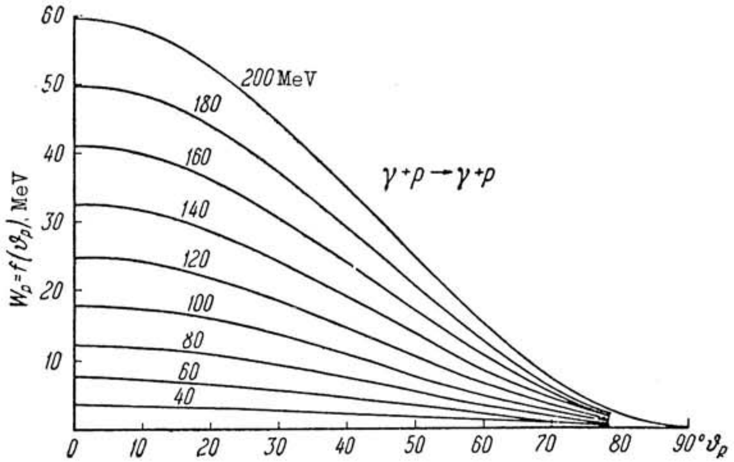

Fig. 11

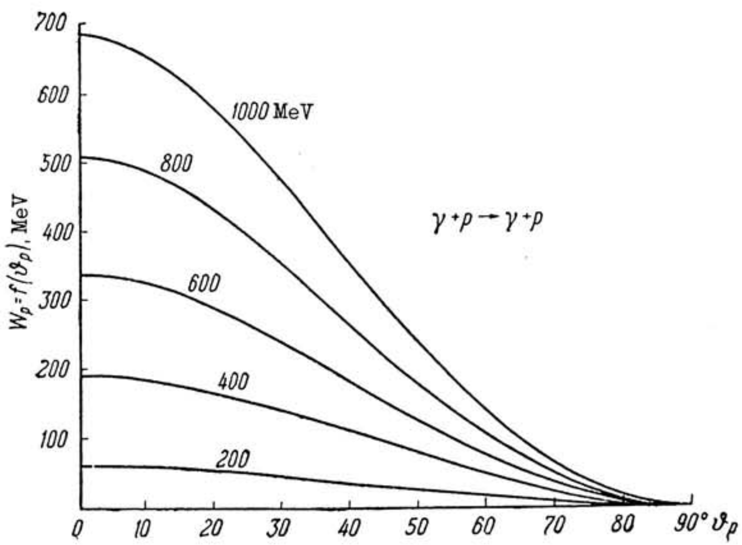

Fig. 12

follow along the dotted arc corresponding to this index, upwards to the intersection with the continuous radial straight line labelled 60°. From the point of intersection we drop down along the continuous arc. The point of its interaction with the horizontal axis of the energy of the neutrons gives $W_n = 4$ MeV.

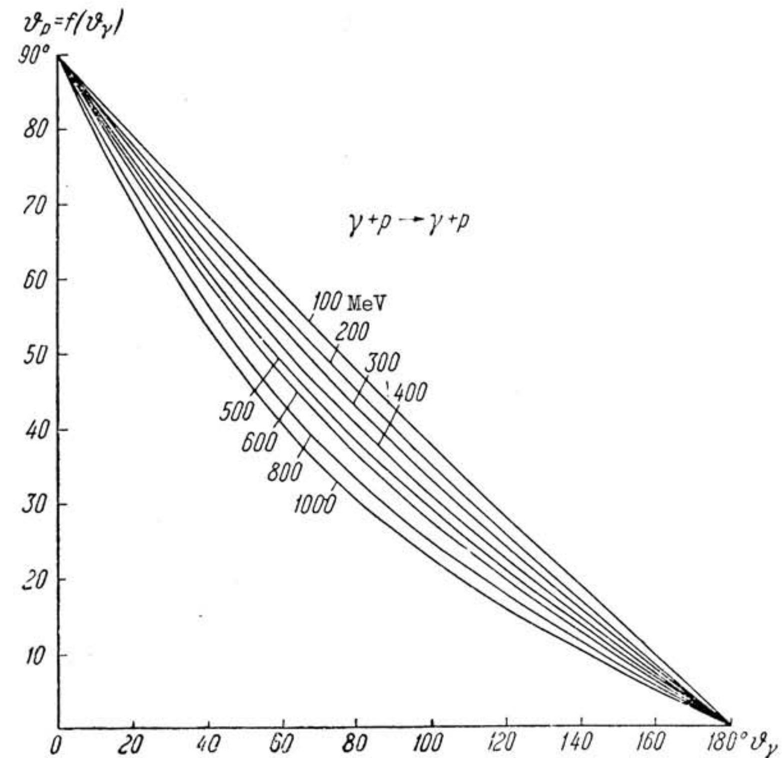

Fig. 13

The subsequent graphs are devoted to a few of the simplest examples of interaction of elementary particles at higher energies. In Figures 9-13, data is shown for the scattering of photons with energies up to 1000 MeV by hydrogen

(Compton effect by a proton) - in the first two graphs, the relationship between the energies of the photons and the angles of their scattering is shown, and in the two subsequent graphs the relationships between the energy of the recoil protons and their angle of emission, and finally, in Figure 13, the relationship between the angles of emission of the scattered photons and the recoil protons. The numbers by the curves in these figures represent the energy of the primary photon in MeV. It is obvious that the graphs in Figures 9-13 are also applicable to the Compton scattering of electrons, if all values of energy of the photons and of the recoil protons are decreased by a factor of $\frac{M}{m} \approx$ 1839.

In Figures 14-18, similar data for the photoproduction of π^+ -mesons at protons is depicted: $\gamma + p \rightarrow \pi^+ + n$. These graphs can also be used for kinematic analysis of other meson photoproduction reactions ($\gamma + n \rightarrow \pi^- + p$ and $\gamma + N \rightarrow \pi^0 + N$; where N is the symbol for a nucleon). Figures 14 and 15 show the relationship between the kinetic energy of π -mesons and their angles of emission for $h\nu = E_\gamma = 160$ - 1000 MeV. A similar relationship between the energy of the nucleons and their angles of emission is given in Figures 16 and 17. Figure 18 shows the relationship between the angles of emission of π-mesons and nucleons in processes of photoproduction of π -mesons.

The subsequent three figures (Figures 19-21) are devoted to the kinematics of the photodisintegration of the deuteron for photon energies within the range 5 to 1000 MeV.

Figures 22-30 illustrate the elastic scattering of nucleons by nucleons (for example pp-scattering - Figures 22-25) and elastic scattering of π -mesons by nucleons. Data for the pp -scattering is given for proton energies from 20 MeV to 10 BeV, data on πN-scattering for π -mesons with energies from 20 MeV to 1 BeV. As in the previous examples, the relationships between the energies of each of the secondary particles and its angle of emission are given, and also the relationship between the angles of emission of the two particles. The numbers by the curves in these figures denote the energy of the incident particle.

In Figures 31 and 32* the kinematic relationships for the

*The graphs in Figures 9-32 have been specially constructed for this work by B.B. Govorkov.

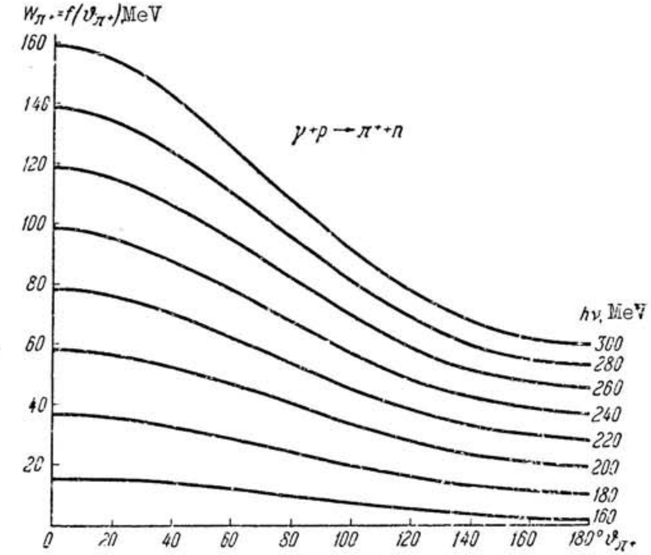

Fig. 14

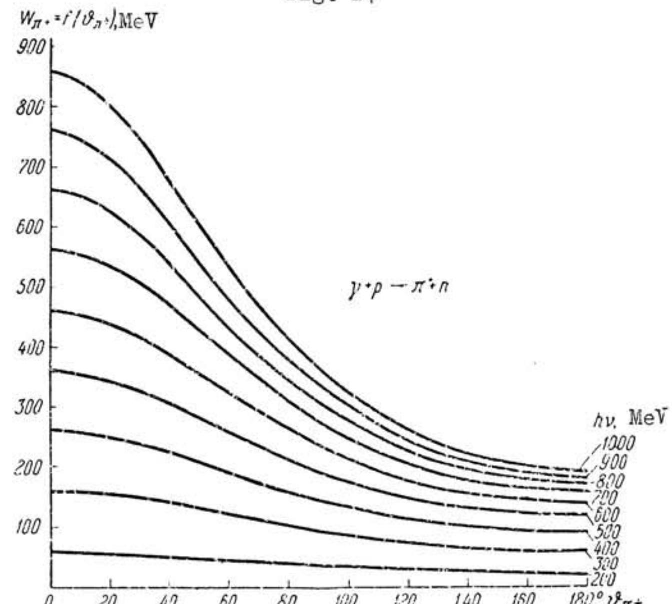

Fig. 15

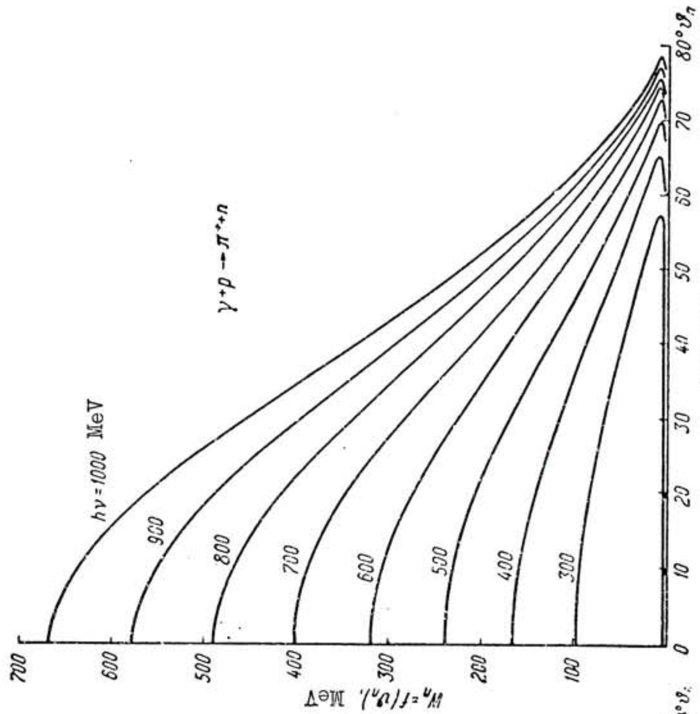

Fig. 17

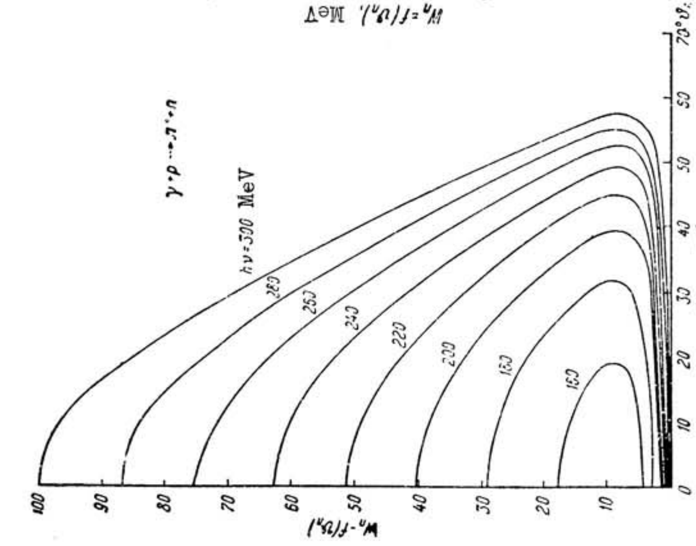

Fig. 16

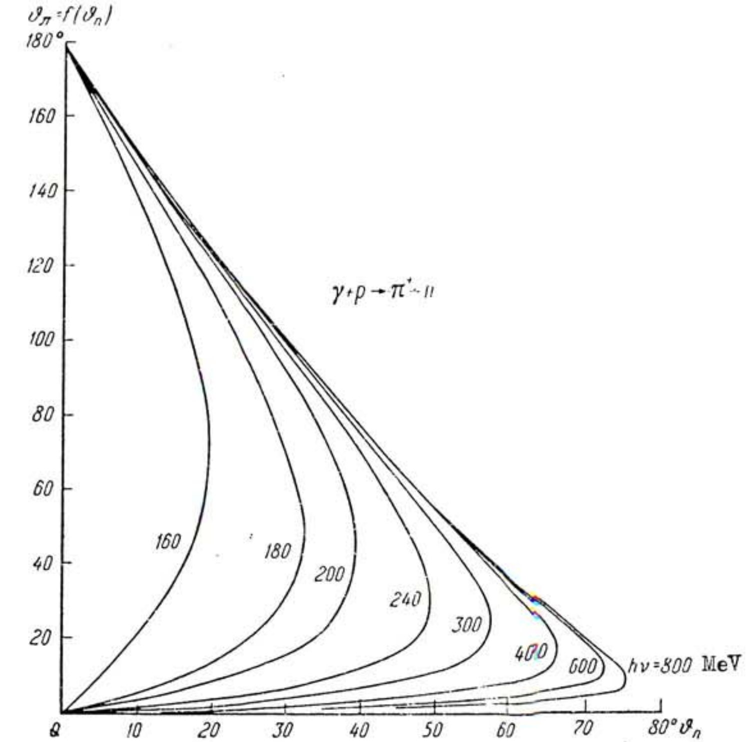

Fig. 18

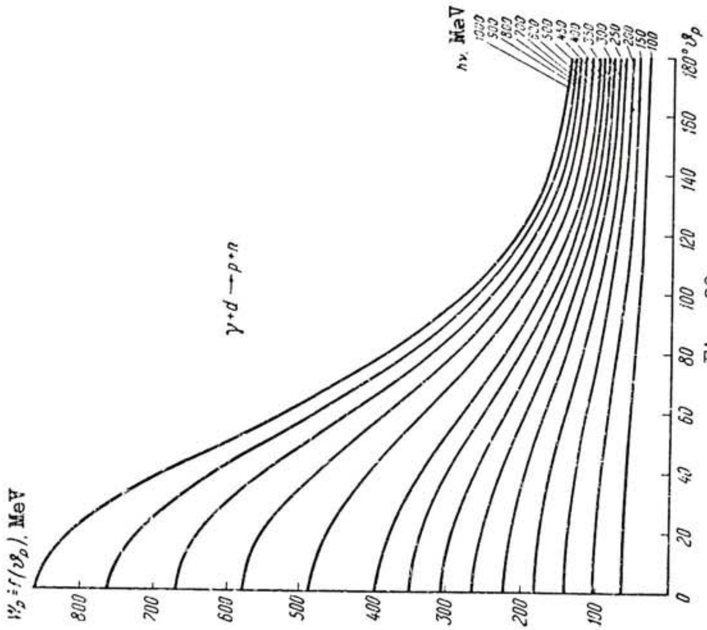

Fig. 20

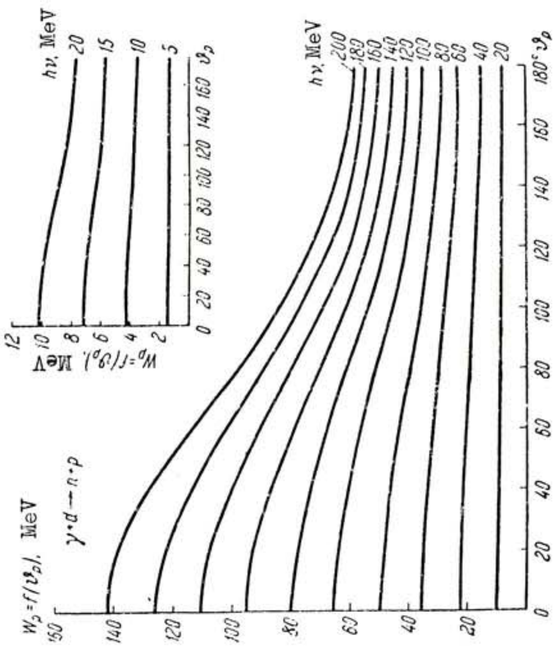

Fig. 19

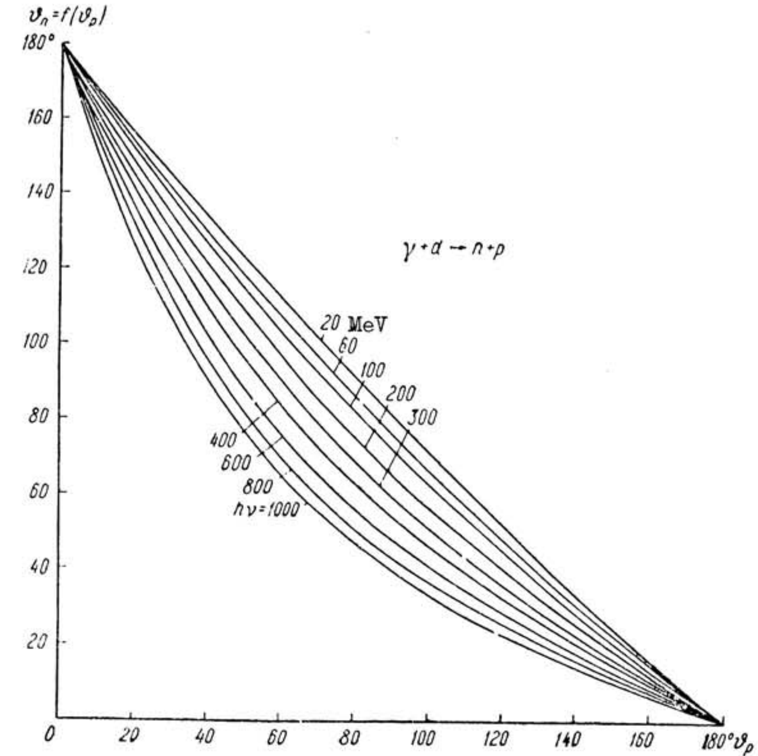

Fig. 21

reaction $\pi^+ + d \to p + p$ for π^+-mesons with energies of 50-500 MeV are given.

The subsequent six figures (Figures 33-38) taken from a paper by Perdetti (Perdetti, Nuovo Cimento, III, No. 5,

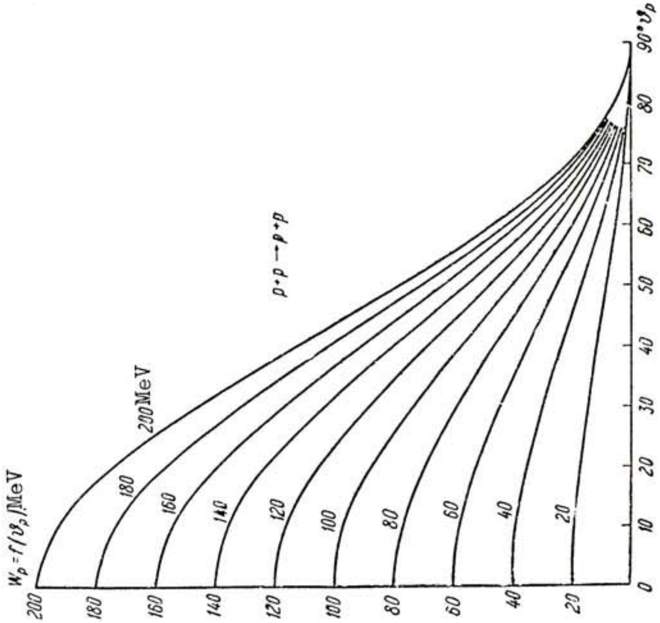

Fig. 22

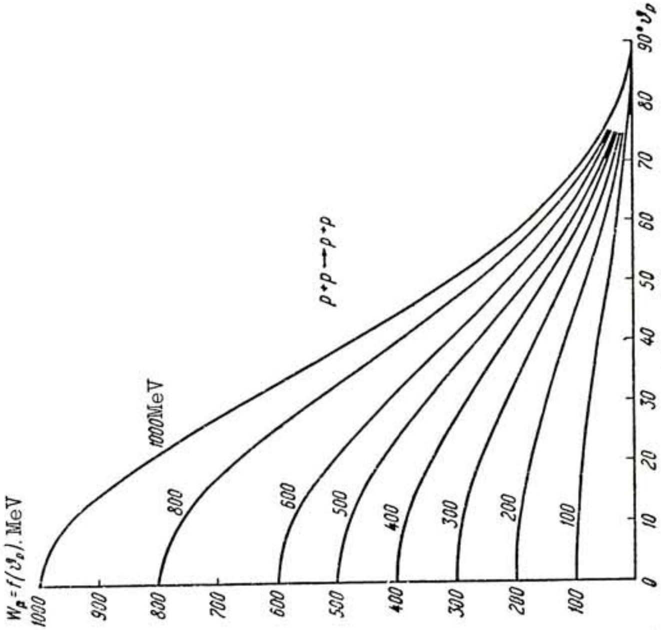

Fig. 23

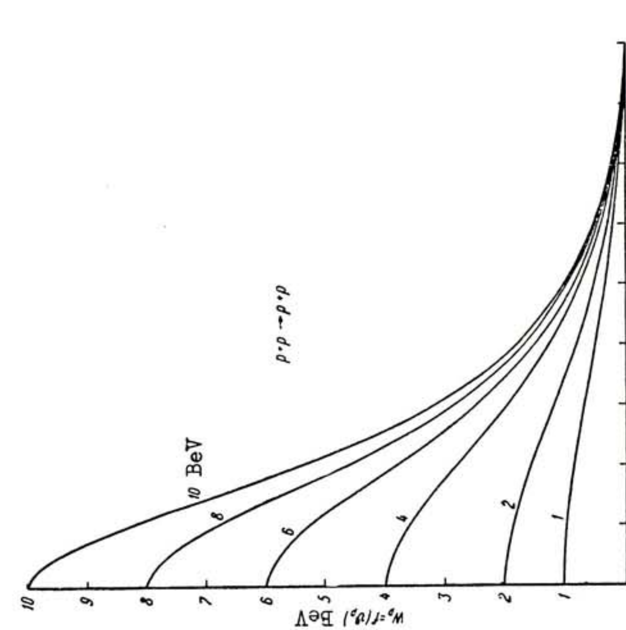

Fig. 24

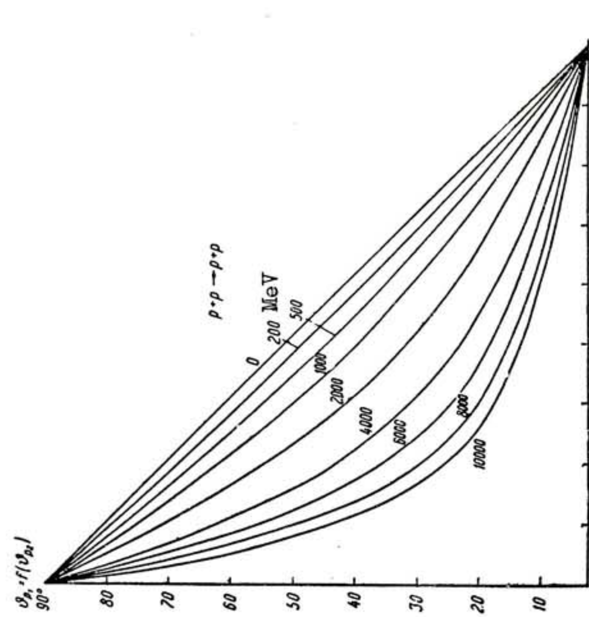

Fig. 25

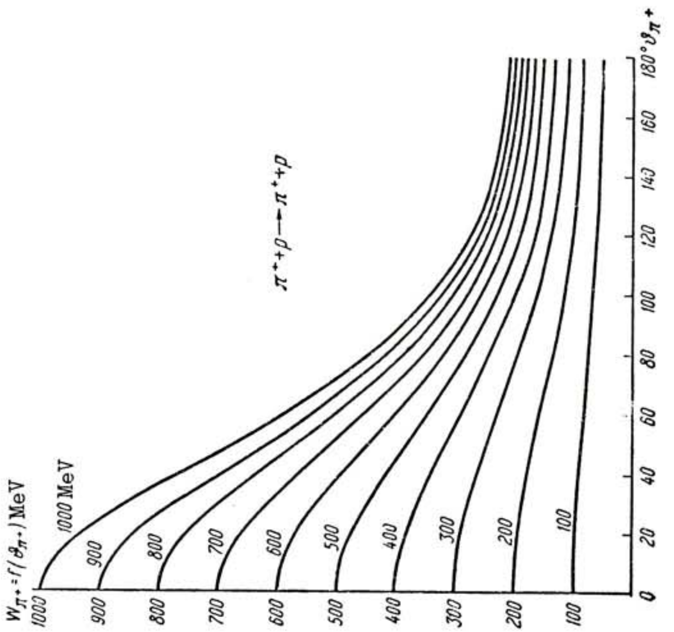

Fig. 27

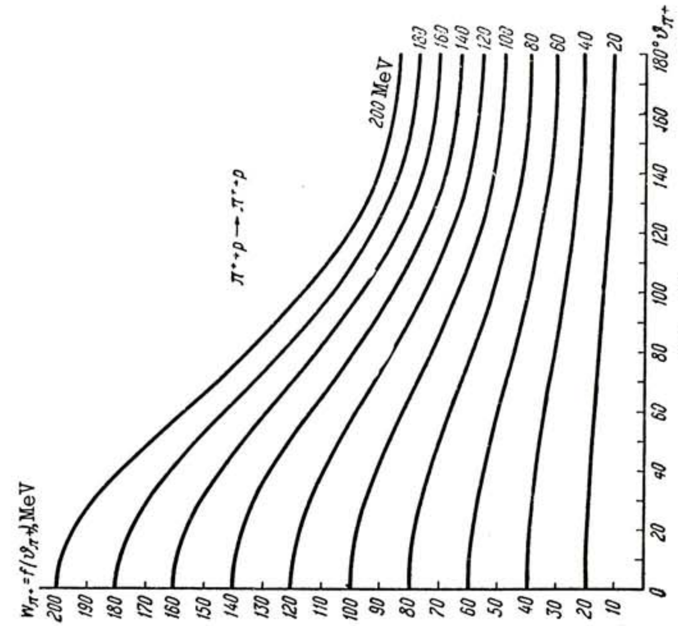

Fig. 26

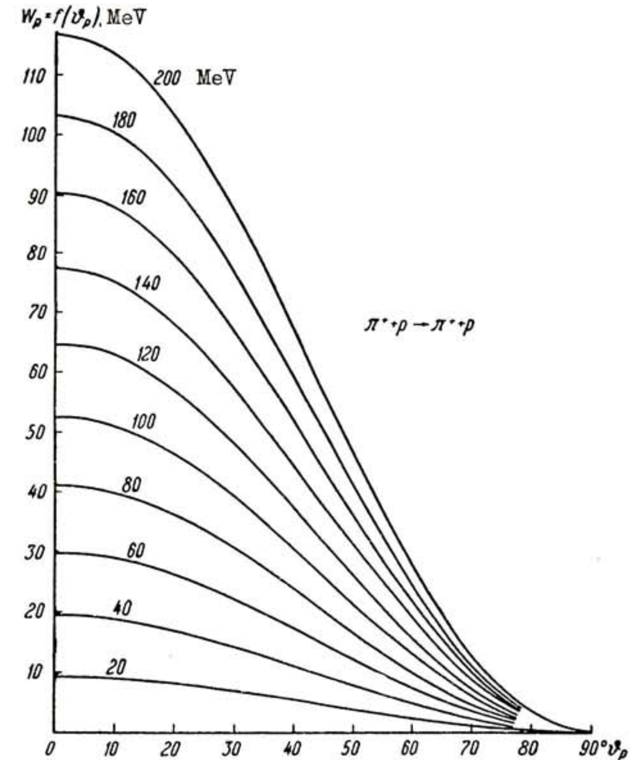

Fig. 28

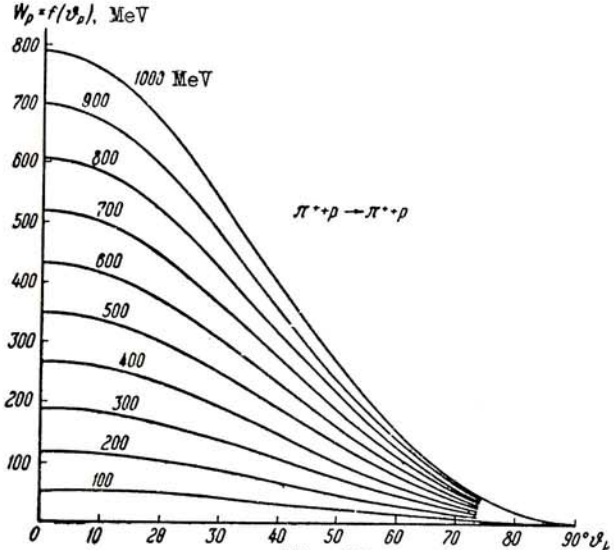

Fig. 29

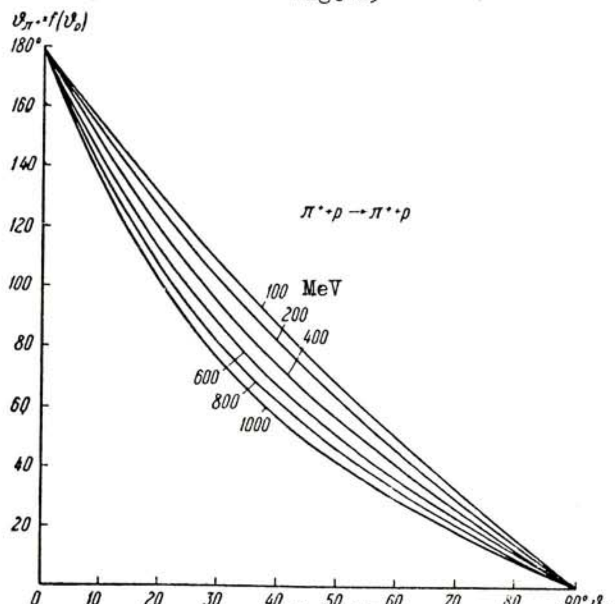

Fig. 30

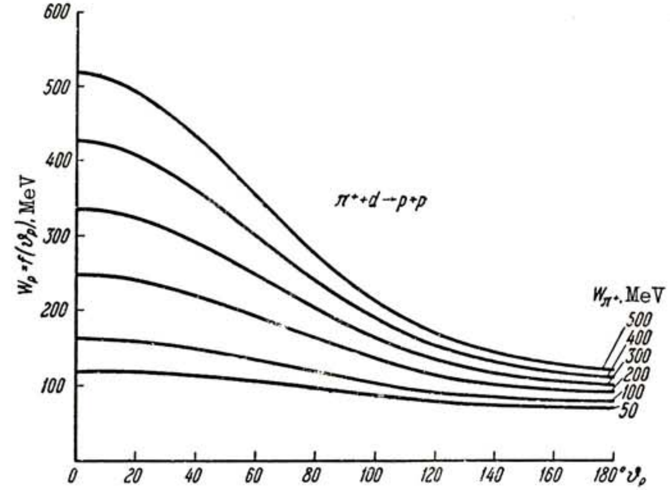

Fig. 31

(1956) relate to the interaction of K-mesons with protons*).

Figures 33-35 show the kinematic characteristics of elastic scattering, $K+p \to K+p$, viz. - the relationship between the angles of emission of the K-mesons and protons in the L-system, and the corresponding angles in the CM-system, for K-meson energies in the L-system of 50, 125 and 200 MeV (Figure 33) and the relationship between the kinetic energy of the K-mesons and protons, and their angles of emission in the L-system (Figure 34, 35) for an initial energy of 10-200 MeV.

*The kinematics of interaction of K-mesons with nucleons are given in tables published in 1958 in the form of a Supplement to the Journal "Nuovo Cimento (VIII, Suppl. No. 1, 1958). The kinematics of the reaction $K^- + p \to \Sigma^\pm + \pi^\mp$ and of elastic scattering $K+p \to K+p$ is discussed in them. The relationship is given between the angles and energies in the L-system for both the products of the reaction for different angles of emission of the π- and K-mesons in the CM-system. The range of energy of the incident K-mesons in these tables is from 5 to 200 MeV (in 5 MeV intervals), and of the angles of emission $\tilde{\vartheta}$, from 0 to 180° (by 2.5°).

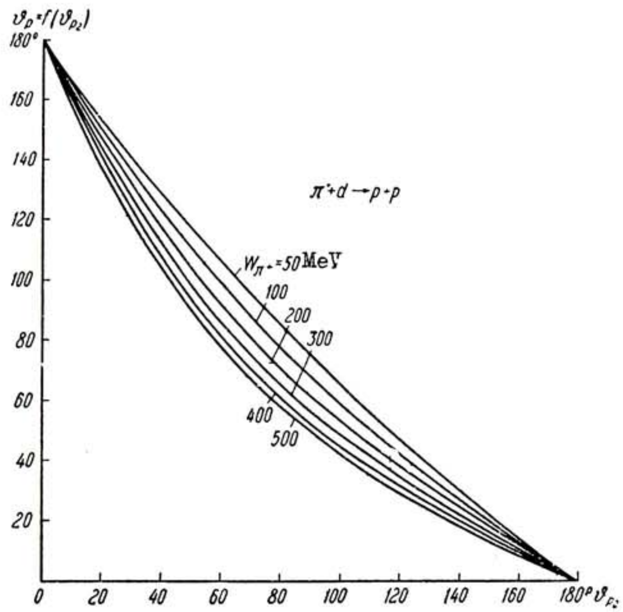

Fig. 32

Figures 36-38 illustrate the kinematics of the reaction $K + p \to \pi + \Sigma$ for the K-mesons with energies from 60-160 MeV. Figures 36 and 37 are analagous to Figures 33 and 34, and in Figure 38 the kinetic energy of the Σ-hyperon is given as a function of the angles of emission, not of this particle, but of the other reaction product, the π-meson.

Finally, Figure 39, taken from a paper by James and Salmeron (G. James and R. Salmeron, Phil. Mag., 46, 576, 1955), gives the relationship between the angle of emission of Λ-particles (upper) and K-mesons (lower) in the reaction $\pi + N \to K + \Lambda$ and the recoil momentum of these particles. The range of momenta of the primary π-mesons in this figure is from 1 to 10 BeV (numbers by the continuous curves). The dotted lines in this figure denote different values of the angles of emission of the secondary particles in the CM-system.

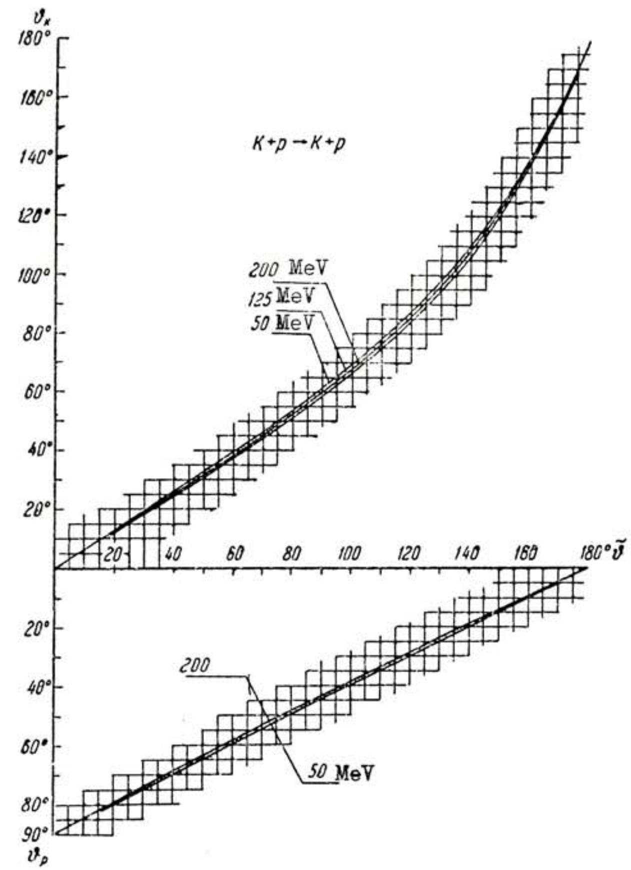

Fig. 33

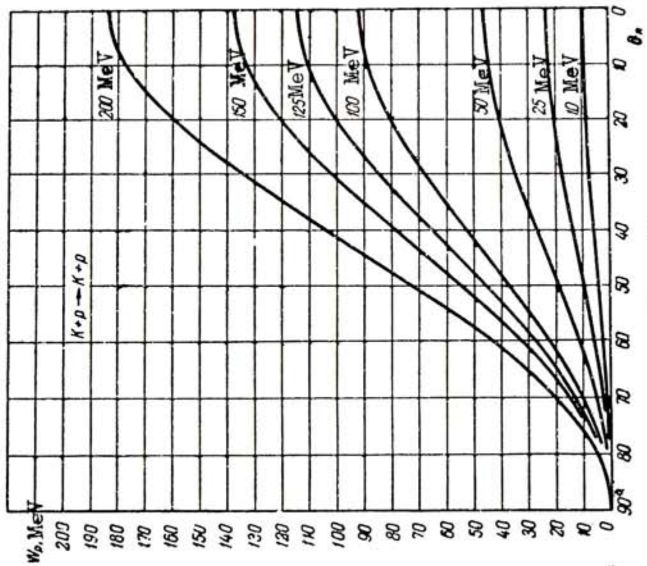

Fig. 35

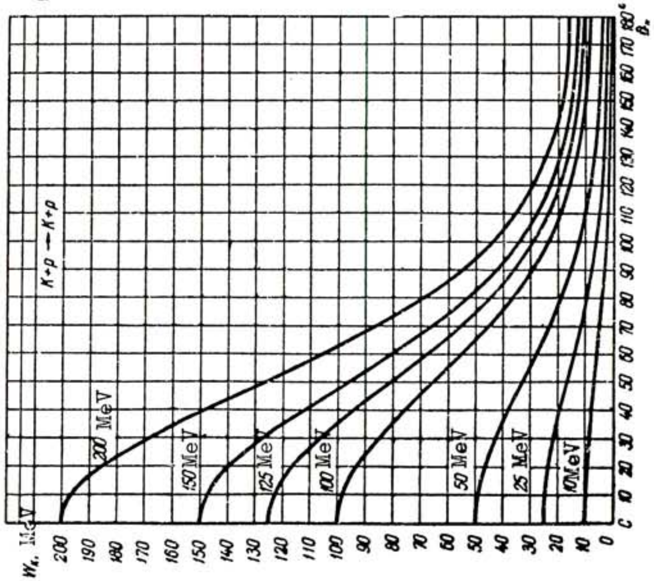

Fig. 34

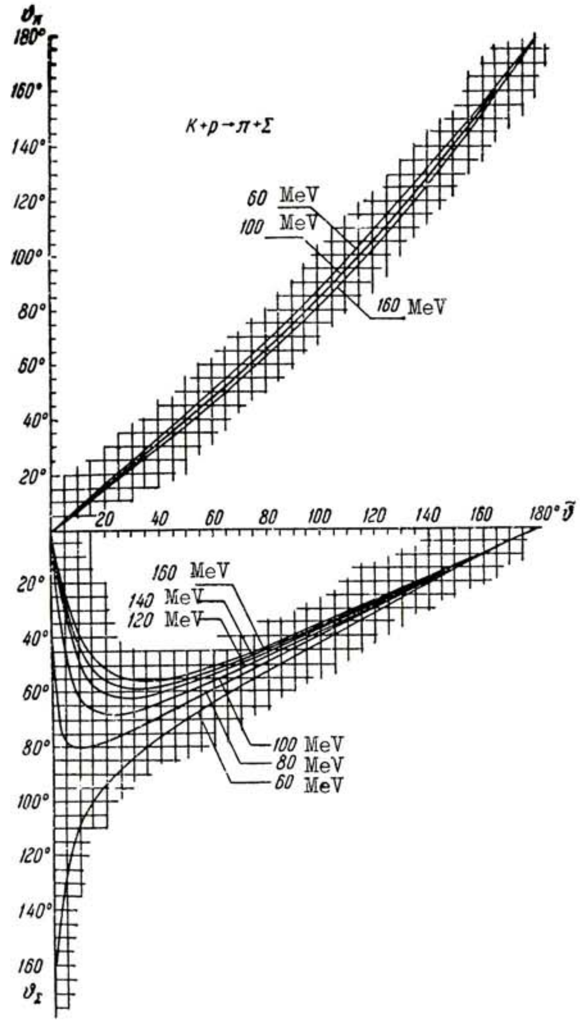

Fig. 36

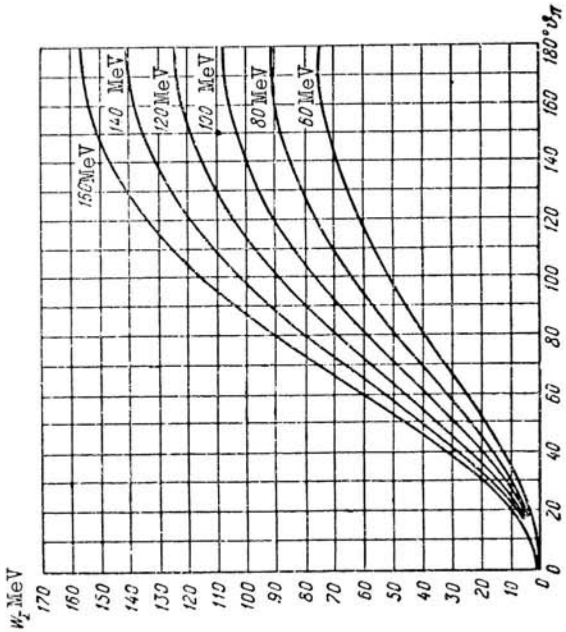

Fig. 38

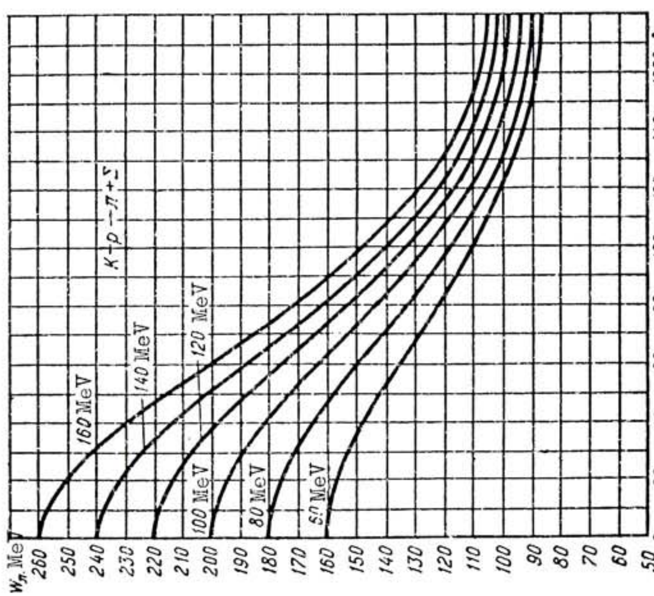

Fig. 37

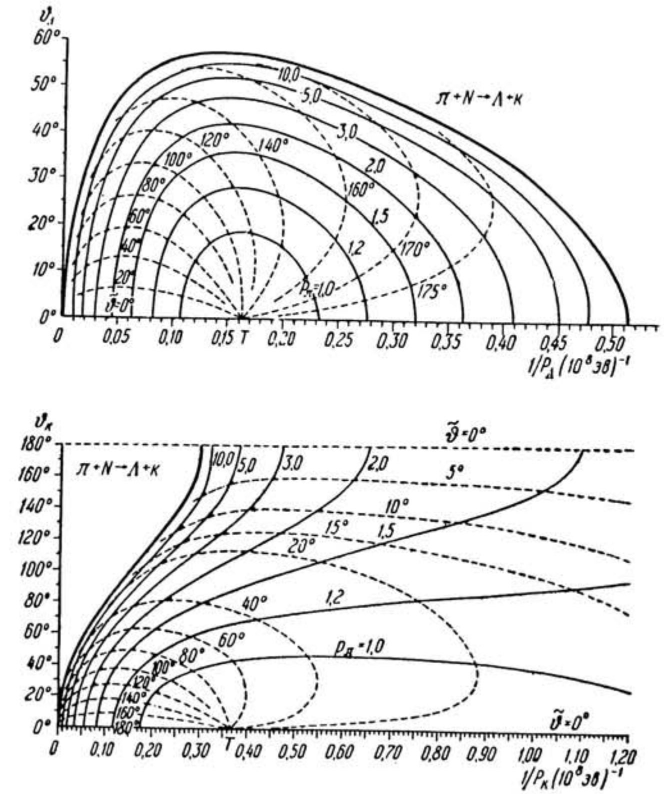

Fig. 39

In conclusion, it is necessary to reiterate that our book has no pretence to being a complete handbook on the kinematics of nuclear interactions, and therefore the examples presented in the Appendix by no means exhaust all that is essential in daily practical work with cosmic radiation or in accelerators. However, the essential expressions can be easily obtained in all cases by means of the formulae presented in the main text.

APPENDIX II*

(To Part Two)

Tables of W, Z, Z_γ and X-Coefficients

a) Introductory Remarks

Tables of W, Z, Z_γ and X-coefficients are given below together with brief explanations of them, concerning changes of argument and the accepted methods of recording the numerical value of the coefficient. All the essential information associated with the determination and properties of the W, Z, Z_γ and X-coefficients is given in the text of the book.

The numerical tables introduced here have been compiled from the basic works [30, 36, 37, 38]. The tables of Z_γ coefficients are newly complied. All the tables have been carefully compared, whenever possible, with existing tables. In doubtful cases the coefficients were re-calculated.

In selecting the tables, we set ourselves the problem of deducing the values of the coefficients for the most important, and at the same time the simplest, cases of nuclear reactions, which nevertheless should embrance a wide circle of phenomena (nuclear reactions with spin of the channel not exceeding $^3/_2$, photonuclear reactions and scattering of photons by particles with a total spin not exceeding $^3/_2$, photoproduction of mesons by particles with a spin of $^1/_2$ etc). Unfortunately, we were not able to provide these tables, mainly because of their considerable bulk, for the whole of the material in the book. In a number of cases (this principally, relates to polarized particles in nuclear reactions, and to reactions involving photons) by using the general formulae of the main text, the student is able to take advantage of the more complete tables of the required

*This Appendix has been compiled by A.I. Lebedev and V.A. Petryn'kin.

coefficients*, or to calculate them for himself. For facilitating the latter problem, in addition to the basic tables of the W, Z, Z_γ and X-coefficients, certain supplementary tables and formulae are given in this Appendix. Here are given tables of vector addition coefficients $(l_1 0 l_2 0 | L0)$ and $(l_1 -1 l_2 0 | L0)$, which are necessary for calculating the Z and Z_γ coefficients according to formulae (29.17) and (29.20) of the main text, tables of factorials of numbers, tables of formulae for calculating vector addition coefficients and W-coefficients. In the supplementary tables are also tabulated the values of $\Delta(abc)$, which is determined by the relationship

$$\Delta(abc) = \left[\frac{(a+b-c)!\,(b+c-a)!\,(c+a-b)!}{(a+b+c+1)!}\right]^{\frac{1}{2}},$$

whereupon a, b, and c satisfy the condition of triangulation, and their sum is a whole number. Obviously, the value of $\Delta(abc)$ is symmetrical with respect to any permutation of its argument. With its aid, the explicit form of the W-coefficient can be written down:

$$W(abcd;\, ef) = \Delta(abe)\,\Delta(cde)\,\Delta(acf)\,\Delta(bdf)\,\omega(abcd;\, ef),$$

$$\omega(abcd;\, ef) =$$

$$= \sum_z \frac{(-1)^{a+b+c+d+z}(z+1)!}{(z-a-b-e)!\,(z-c-d-e)!\,(z-a-c-f)!\,(z-b-d-f)!} \times$$
$$\times \frac{1}{(a+b+c+d-z)!\,(a+d+e+f-z)!\,(b+c+e+f-z)!},$$

*From the most complete tables of W and Z-coefficients, we draw attention first and foremost to the voluminous tables compiled in the Oak Ridge Laboratories [30]. Tables of W, Z and X-coefficients are given in reports from the Chalk River Laboratories [37]; these tables, unfortunately, are prepared primarily for analysis of correlations in nuclear reactions. The range of variation of the variables is different from the ones presented here. The W and X-coefficients, for a fairly wide range of variation of arguments, can be found respectively in works [38, 36], by Japanese authors. Coefficients approximately to Z_γ are tabulated in a synopsis by Biedenharn and Rose [27] and in reports from the Chalk River Laboratories [37].

where z takes only those integral values which do not lead to a negative argument for the factorials in the denominator. The vector addition coefficients $(a0b0\,|\,c0)$ for the integrals a, b, c with an even sum $a+b+c=2g$ are likewise expressed by $\Delta(abc)$:

$$(a0b0|c0) = (-1)^{g+c}\cdot(2c+1)^{\frac{1}{2}}\,\Delta(abc)\,\frac{g!}{(g-a)!\,(g-b)!\,(g-c)!}.$$

For a, b, c, not satisfying this condition, the coefficient $(a0b0\,|\,c0)$ vanishes to zero. The value of $[\Delta(abc)]^{-2}$ is given in the tables.

Finally, in order to facilitate calculation of the different angular distributions, we give in paragraph e) the explicit form of the Legendre polynominals $P_L(\cos\theta)$ and the normalize of associated Legendre functions with $M=1$ $\overline{P_L^M}(\cos\theta)$, which enter respectively into expressions for differential cross-sections and polarization.

b) Layout of Tables and Range of Variation of Arguments

All the coefficients are given in the form of fifty-seven tables.

The first 15 tables (I to XV inclusive) contain the coefficients $W^2(l_1J_1l_2J_2;\ sL)$, for which s assumes the values $1/2$, 1 and $3/2$, and L takes all integral values from 0 to 4. In the tables with $s=1/2, 3/2$, l_1 and l_2 assume integral values from 0 to 5, and J_1 and J_2 vary from $1/2$ to $9/2$. In the tables with $s=1$ l_1 and l_2 assume integral values from 0 to 5, J_1 and J_2 vary from 0 to 4.

The tables of coefficients $Z^2(l_1J_1l_2J_2;\ sL)$ have index numbers from XVI to XXXIII. The range of variation of the variables is, as in the case of the W- coefficients, with the exception that L takes the additional value $L=5$.

The coefficients $Z_\gamma^2(l_1J_1l_2J_2;\ sL)$, which are necessary for analysis of photonuclear reactions, are calculated only for the cases $s=1/2$. They are contained in Tables XXXIV-XXXVIII, in which l_1, l_2 assume integral values from 0 to 3 and J_1, J_2 vary from $1/2$ to $5/2$. L takes all integral values from 0 to 4.

The W, Z and Z_γ-coefficients are presented in the form of individual tables for each value of s and L, whereupon for l_1, l_2, J_1 and J_2 values are given for which the corresponding coefficients are determined. The tables are arranged in increasing order of L (for a given s), and then s. We do not give in the tables the W, Z and Z_γ-coefficients for $s=0$, since the reversion of s to zero essentially simplifies the general expressions for these coefficients (see formulae (29.10), (29.18) and (29.21) of the main text), and the reader can calculate them for himself without much expenditure of effort.

The values of the coefficients $X^2(abc,\ def,\ ghk)$ are presented in the form of 6 tables. In the first four tables, XXXIX to XLII, $g=h, c=f=1/2$, $k=1$, a and d take semi-integral values from $1/2$ to $7/2$, b and e take integral numbers from 0 to 4. The tables are arranged in increasing order of g, which changes from 1 to 4, assuming integral values. Tables XLIII and XLIV, respectively, contain the coefficients $X^2(ab1,\ ab2,\ kk1)$ and $X^2(a1c,\ a3c,\ kk1)$, where a and b vary from 1 to 3, assuming integral and semi-integral values, c takes all integral and semi-integral values from 1 to $5/2$, k takes the values 2 and 4 .

Supplementary tables are also given of the values of $[\Delta(abc)]^{-2}$. In Table XLV, a, b and c are integral numbers, where a and b assume values from 1 to 8, and c varies from 0 to 6. In Table XLVI, a and b are semi-integral and vary within the limits from $1/2$ to $11/2$; c takes all possible values. The vector addition coefficients $(l_10l_20\,|\,L0)^2$ for l_1, l_2 assuming the values from 0 to 6, are given in Table XLVII. Table XLVIII contains the coefficients $(l_1-1l_21\,|\,L0)^2$ for l_1 and l_2 equal to 1, 2 and 3. In Table XLIX, factorials of the numbers from 1 to 24 inclusive are given.

Occasionally, in order to obtain the values of the coefficients X, $(l_10l_20\,|\,L0)$, $(l_1-1l_21|L0)$ and the values of $\Delta(abc)$ which are not given in the tables, it is sufficient to use the properties of symmetry of these coefficients.

Finally, Tables L-LIII and LIV-LVII, respectively, contain formulae for calculating the vector addition coefficients $(j_1m_1j_2m_2\,|\,jm)$ and the coefficients $W(l_1J_1l_2J_2,\ sL)$, for j_2 and s taking the values $1/2$, 1, $3/2$ and 2.

c) Method of Recording the Numerical Values of the Coefficients

We draw the attention of the student to the fact that in all the calculated tables, with the exception of the table of factorials of numbers, the squares of the values of tne coefficients are given. If the same coefficient is negative in value, then in front of its square stands the sign ∗ .

The calculated values of the squares of the coefficients are rational fractions which, with rare exception for the limits considered, changes of argument amount to only simple multiples, are not greater than 19. Usually, in the Appendices, it is necessary to multiply certain of such rational fractions in order to find the calculated coefficient of the Legendre polynominals $P_L(\cos\theta)$ or for normalization of the associated Legendre functions $\overline{P_L^1}(\cos\theta)$. This operation is simplified if, in place of the fraction, only the exponent of the powers of those simple numbers is written, in which the numerator and denominator of the latter is expanded. Exponents of powers of simple numbers are written down in the following order: first of all the power of two is written, secondly the power of three, thirdly the power of five etc. If the simple number, next in order of sequence, is absent in the breakdown, then in its place is written zero. Negative powers of simple numbers are recorded by underlining. For more rapid orientation, exponents of powers of simple numbers of the first ten are distinguished by a point from the remaining exponents. If in the breakdown a simple number is encountered greater than 19, then it is written down to the right in parentheses in an explicit form in the corresponding power. If the exponent exceeds ten, then only the amount of the excess is written together with a line above it. As a unit, the notation in the form of e is used. Such an entry we shall call the representation of the number. For example, the representation of the number 30 will be 111, since $30 = 2^1 \cdot 3^1 \cdot 5^1 \to 111$. Let us give a further series of examples:

$$198 = 2^1 \cdot 3^3 \cdot 5^0 \cdot 7^0 \cdot 11^1 \longrightarrow 1300.1$$

$$65/7056 = 2^{-4} \cdot 3^{-2} \cdot 5^1 \cdot 7^{-2} \cdot 11^0 \cdot 13^1 \longrightarrow \underline{4}\underline{2}1\underline{2}.01$$

$$1 \longrightarrow e$$

$$6144 = 2^{11} \cdot 3^1 \longrightarrow \bar{1}1$$

$$50410/17787 = 2^1 \cdot 3^{-1} \cdot 5^1 \cdot 7^{-2} \cdot 11^{-2} \cdot (71)^2 \longrightarrow 1\underline{1}1\underline{2}.\underline{2} \cdot (71)^2.$$

Multiplication of number is accomplished by means of addition of their representations:

$$1260 \cdot 1/7 \cdot 1/210 \longrightarrow 2211 + 000\underline{1} + \underline{1111} = + \begin{array}{l} 2211 \\ 000\underline{1} \\ \underline{\underline{1111}} \\ \hline 110\underline{1} \longrightarrow 6/7 \end{array}$$

$$2/7 \cdot 252 \cdot 1/18 \longrightarrow 100\underline{1} + 2201 + \underline{12} = + \begin{array}{l} 2201 \\ 100\underline{1} \\ \underline{\underline{12}} \\ \hline 2 \longrightarrow 4 \end{array}$$

Blank compartments and omissions in the tables denote that the corresponding coefficients are either non-existent (i.e. some conditions for determiration of the coefficient are violated), or are equal to zero.

d) Brief Instructions for Obtaining the Numerical Value of a Coefficient from the Tables

In order to obtain the numerical value of a coefficient itself, the number required from the table is transposed into a general representation in accordance with the rule $abcd, e\ldots = 2^a \cdot 3^{-b} \cdot 5^{10+c} \cdot 7^d \cdot 11^e \ldots$, and from the result obtained the square root is extracted; if against the number in the table stands the sign ∗ , then the negative value is taken for the root.

e) Legendre Functions

For the convenience of students we are giving here a few formulae for functions by which various angular distributions in nuclear reactions are expressed.

Normalization of a spherical harmonic is obtained by the relationship

$$Y_{LM}(\theta, \varphi) = \overline{P_L^M}(\cos\theta)\frac{1}{\sqrt{2\pi}}e^{iM\varphi},$$

where $\overline{P_L^M}(x)$ are the normalized associated Legendre functions. They are related in the following manner with $P_L^M(x)$:

$$\overline{P_L^M}(x) = (-1)^M\sqrt{\frac{2L+1}{2}\frac{(L-M)!}{(L+M)!}}P_L^M(x),$$

which, in turn, are determined by the formula

$$P_L^M(x) = (1-x^2)^{\frac{M}{2}}\frac{1}{2^L L!}\frac{d^{L+M}}{dx^{L+M}}(x^2-1)^L.$$

For $M>0$ $P_L^M(x)$ are expressed via the general Legendre polynominals $P_L(x)$

$$P_L^M(x) = (1-x^2)^{\frac{M}{2}} \frac{d^M}{dx^M} P_L(x),$$

whereupon

$$P_L(x) = \frac{1}{2^L L!} \frac{d^L}{dx^L} (x^2-1)^L.$$

We draw attention to the choice of phase, made as a result of the determination of $\overline{P_L^M}(x)$, for the normalized factor. In the general theory of spherical functions, the normalized factor does not contain $(-1)^M$. Here we have related $(-1)^M$ to $\overline{P_L^M}(\cos\theta)$ so that our determination of the spherical harmonic agrees with Wigner's determination [12]. $\overline{P_L^M}(x)$ satisfies the relationship

$$\overline{P_L^M}(x) = (-1)^M \overline{P_L^{-M}}(x),$$

whence

$$Y^*_{LM}(\theta, \varphi) = (-1)^M Y_{L,-M}(\theta, \varphi).$$

We are writing out the explicit form of $P_L(x)$ and $\overline{P_L^1}(x)$ for L, taking integral values from 0 to 4 inclusive ($x=\cos\theta$):

$$P_0(x) = 1,$$

$$P_1(x) = x = \cos\theta,$$

$$P_2(x) = \frac{1}{2}(3x^2-1) = \frac{1}{4}(3\cos 2\theta + 1),$$

$$P_3(x) = \frac{1}{2}(5x^3-3x) = \frac{1}{8}(5\cos 5 3 + 3\cos\theta),$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) = \frac{1}{64}(35\cos 4\theta + 20\cos 2\theta + 9),$$

$$\overline{P_1^1}(x) = -\sqrt{\frac{3}{4}}(1-x^2)^{\frac{1}{2}} = -\sqrt{\frac{3}{4}}\sin\theta,$$

$$\overline{P_2^1}(x) = -\sqrt{\frac{15}{4}}(1-x^2)^{\frac{1}{2}}x = -\sqrt{\frac{15}{16}}\sin 2\theta,$$

$$\overline{P_3^1}(x) = -\sqrt{\frac{21}{32}}(1-x^2)^{\frac{1}{2}}(5x^2-1) = -\sqrt{\frac{21}{512}}(5\sin 3\theta + \sin\theta),$$

$$\overline{P_4^1}(x) = -\sqrt{\frac{45}{32}}(1-x^2)^{\frac{1}{2}}(7x^3-3x) =$$

$$= -\sqrt{\frac{45}{2048}}(7\sin 4\theta + 2\sin 2\theta).$$

TABLE I

$W^2\left(l_1 J_1 l_2 J_2;\ \frac{1}{2} 0\right)$

J_2	l_2 \ J_1, l_1	1/2: 0	1/2: 1	3/2: 1	3/2: 2	5/2: 2	5/2: 3	7/2: 3	7/2: 4	9/2: 4	9/2: 5
1/2	0	1									
	1		*11								
3/2	1			21							
	2				*201						
5/2	2					111					
	3						*1101				
7/2	3							3001			
	4								*32		
9/2	4									121	
	5										*1010.1

TABLE II

$$W^2\left(l_1 J_1 l_2 J_2;\ \frac{1}{2}\ 1\right)$$

J_2	J_1 → / l_2 \ l_1	1/2: 0	1/2: 1	3/2: 1	3/2: 2	5/2: 2	5/2: 3	7/2: 3	7/2: 4	9/2: 4	9/2: 5
1/2	0		11	11							
	1	11	02	*22	*21						
3/2	1	11	*22	*321	311	201					
	2		*21	311	322	*212	*111				
5/2	2			201	*212	*0221	1211	1101			
	3				*111	1211	1212	*3102	*3001		
7/2	3					1101	*3102	*5302	5201	32	
	4						*3001	5201	5411	*341	*121
9/2	4							32	*341	*1420.1	1220.1
	5								*121	1220.1	0320.2

TABLE III

$$W^2\left(l_1 J_1 l_2 J_2;\ \frac{1}{2}\ 2\right)$$

J_2	J_1 → / l_2 \ l_1	1/2: 0	1/2: 1	3/2: 1	3/2: 2	5/2: 2	5/2: 3	7/2: 3	7/2: 4	9/2: 4	9/2: 5
1/2	0				101	101					
	1			21	201	*021	*12				
3/2	1		21	31	*301	*2211	1201	2001			
	2	101	201	*301	*3021	202	0011	*3011	*301		
5/2	2	101	*021	*2211	202	112	*1111	*2111	231	13	
	3		*12	1201	0011	*1111	*0002	3002	3321	*0311	*1011
7/2	3			2001	*3011	*2111	3002	5022	*5101	*3111.1	2011.1
	4				*301	231	3321	*5101	*5300.1	331	2111.1
9/2	4					13	*0311	*3111.1	331	0321	*1120.1
	5						*1011	2011.1	2111.1	*1120.1	*1020.21

TABLE IV

$$W^2\left(l_1J_1l_2J_2;\ \frac{1}{2}\,3\right)$$

J_2 \ J_1	l_2 \ l_1	1/2: 0	1/2: 1	3/2: 1	3/2: 2	5/2: 2	5/2: 3	7/2: 3	7/2: 4	9/2: 4	9/2: 5
1/2	0						1001	1001			
	1					12	1201	*3001	*31		
3/2	1				201	021	*2211	4101	42	22	
	2			201	102	*102	*2111	4001	41	*212	*102
5/2	2		12	021	*102	*102	0111	3001	*33	*0320.1	1020.1
	3	1001	1201	*2211	*2111	0111	1112	*2002	*2301.1	1301	0001,1
7/2	3	1001	*3001	4101	4001	3001	*2002	*4002.1	4101	2101	*3001.1
	4		*31	42	41	*33	*2301.1	4101	431	*231	*3110.11
9/2	4			22	*212	*0320.1	1301	2101	*231	*1320,01	0120.1
	5				*102	1020.1	0001.1	*3001.1	*3110,11	0120,1	1022.2

TABLE V

$$W^2\left(l_1J_1l_2J_2;\ \frac{1}{2}\,4\right)$$

J_2 \ J_1	l_2 \ l_1	1/2: 0	1/2: 1	3/2: 1	3/2: 2	5/2: 2	5/2: 3	7/2: 3	7/2: 4	9/2: 4	9/2: 5
1/2	0								12	12	
	1							31	331	*131	*111
3/2	1						2001	4111	*4301	*2310.1	0110.1
	2					111	2101	*4211	*4210.1	2221	1220.1
5/2	2				111	121	*0201	*3111.1	311	012	*1121,1
	3			2001	2101	*0201	*1202.1	2112	2101	*1111	*0111.11
7/2	3		31	4111	*4211	*3111.1	2112	4202	*4211	*2211.01	3211.1
	4	12	331	*4301	*4210.1	311	2101	*4211	*4400,01	24	3201.1
9/2	4	12	*131	*2310.1	2221	012	*1111	*2211.01	24	1411	*0210,1
	5		111	0110.1	1220.1	*1121.1	*0111.11	3211.1	3201.1	*0210.1	1210.2

TABLE VI

$W^2(l_1J_1l_2J_2;\ 10)$

J_1		0	1			2			3			4		
J_2	l_2 \ l_1	1	0	1	2	1	2	3	2	3	4	3	4	5
0	1	01												
1	0		01											
	1			*02										
	2				011									
2	1					011								
	2						*002							
	3							0011						
3	2								0011					
	3									*0002				
	4										0201			
4	3											0201		
	4												*04	
	5													0200.1

TABLE VII

$W^2(l_1J_1l_2J_2;\ 11)$

J_1		0	1			2			3			4		
J_2	l_2 \ l_1	1	0	1	2	1	2	3	2	3	4	3	4	5
0	1		02	02	02									
1	0	02		02		02								
	1	02	02	22	*22	*22	*201							
	2	02		*22	*201	222	202	002						
2	1		02	*22	222	*201	202		002					
	2			*201	202	202	22	*022	*022	*3211				
	3				002		*022	*3211	0222	0202	0002			
3	2					002	*022	0222	*3211	0202		0002		
	3						*3211	0202	0202	4202.2	*4002	*4002	*4111	
	4							0002		*4002	*4111	4402	44	04
4	3								0002	*4002	4402	*4111	44	
	4									*4111	44	44	4420.0002	*042
	5										04		*042	*3120.1

TABLE VIII

$W^2(l_1J_1l_2J_2;\ 12)$

J_2 \ J_1		0	1			2			3			4		
	l_2 \ l_1	1	0	1	2	1	2	3	2	3	4	3	4	5
0	1					011	011	011						
1	0				011		011	011	011					
	1			22	201	201	221	*021	*021	*1201				
	2		011	201	2121	*202	*2121	*112	0121	0111	0011			
2	1	011		201	*202	2121	*2121	0121	*112	0111		0011		
	2	011	011	221	*2121	*2121	*202	1021	1021	1011	*1111	*1111	*13	
	3	011	011	*021	*112	0121	1021	3122	*0022	*1112	*1112	1212	1301	0201
3	2		011	*021	0121	*112	1021	*0022	3122	*1112	1212	*1112	1301	
	3			*1201	0111	0111	1011	*1112	*1112	*4202	4222	4222	4311.01	*1211
	4				0011		*1111	*1112	1212	4222	4322.1	*4202	*4211.1	*121
4	3					0011	*1111	1212	*1112	4222	*4202	4322.1	*4211.1	1311.1
	4						*13	1301	1301	4311.01	*4211.1	*4211.1	4420.002	1221.1
	5							0201		*1211	*121	1311.1	1221.1	3321.21

TABLE IX

$W^2(l_1J_1l_2J_2;\ 13)$

J_2 \ J_1		0	1			2			3			4		
	l_2 \ l_1	1	0	1	2	1	2	3	2	3	4	3	4	5
0	1								0101	0101	0101			
1	0							0101		0101		0101		
	1						021	1201	1201	3201	*3001	*3001	*331	
	2					002	112	1021	*0111	*3001	*3001	3201	311	021
2	1				002		112	*0111	1021	*3001	3201	*3001	311	
	2			021	112	112		*0121	*0121	*3011	3211	3211	3322	*022
	3		0101	1201	1021	*0111	*0121	*0112	2022	1002	1202.1	*1202	*1311.1	*2211
3	2	0101		1201	*0111	1021	*0121	2022	*0112	1002	*1202	1202.1	*1311.1	0211.1
	3	0101	0101	3201	*3001	*3001	*3011	1002	1002	2002	*2202.1	*2202.1	*2311	1211.1
	4	0101		*3001	*3001	3201	3211	1202.1	*1202	*2202.1	*2312.1	2202	2201	1201.11
4	3		0101	*3001	3201	*3001	3211	*1202	1202.1	*2202.1	2202	*2312.1	2201	*1301.1
	4			*331	311	311	3322	*1311.1	*1311.1	*2311	2201	2201	2422	*1220.11
	5				021		*022	*2211	0211.1	1211.1	1201.11	*1301.1	*1220.11	*0322.21

TABLE X

$W^2(l_1J_1l_2J_2;\ 14)$

J_2	l_2 \ J_1, l_1	0: 1	1: 0	1: 1	1: 2	2: 1	2: 2	2: 3	3: 2	3: 3	3: 4	4: 3	4: 4	4: 5
0	1											03	03	03
1	0										03		03	
	1									3001	331	331	331	*131
	2								0011	3001	3301.1	*311	*3321.1	*012
2	1							0011	-	3001	*311	3301.1	*3321.1	1321.1
	2						222	0201	0201	*3211	*3111.1	*3111.1	*312	0121.1
	3					0011	0201	0212.1	*2202	*1202.1	*1112.1	1102	1121	2121.11
3	2				0011		0201	*2202	0212.1	*1202.1	1102	*1112.1	1121	*0120.1
	3			3001	3001	3001	*3211	*1202.1	*1202.1	*2202	2112	2112	2111	*1111.11
	4		03	331	3301.1	*311	*3111.1	*1112.1	1102	2112	2202.01	*2422	*2401.01	*1400.11
4	3	03		331	*311	3301.1	*3111.1	1102	*1112.1	2112	*2422	*2202.01	*2401.01	1201.1
	4	03	03	331	*3321.1	*3321.1	*312	1121	1121	2111	*2401.01	*2401.01	*24	1401.1
	5	03		*131	*012	1321.1	0121.1	2121.11	*0120.1	*1111.11	*1400.11	1201.1	1401.1	0201.2

TABLE XI

$W^2\left(l_1J_1l_2J_2;\ \frac{3}{2}\,0\right)$

J_2	l_2 \ J_1, l_1	1/2: 1	1/2: 2	3/2: 0	3/2: 1	3/2: 2	3/2: 3	5/2: 1	5/2: 2	5/2: 3	5/2: 4	7/2: 2	7/2: 3	7/2: 4	7/2: 5	9/2: 3	9/2: 4	9/2: 5
1/2	1	11																
	2		*101															
3/2	0			2														
	1				*21													
	2					201												
	3						*2001											
5/2	1							12										
	2								*111									
	3									1101								
	4										*13							
7/2	2											301						
	3												*3001					
	4													32				
	5														*3000.1			
9/2	3															1011		
	4																*121	
	5																	1010.1

TABLE XII

$W^2\left(l_1J_1l_2J_2;\ \frac{3}{2}\ 1\right)$

J_2	J_1 →	1/2		3/2				5/2				7/2				9/2		
	l_2 \ l_1	1	2	0	1	2	3	1	2	3	4	2	3	4	5	3	4	5
1/2	1	22	21	21	321	31												
	2	21	201		*311	*301	*201											
3/2	0	21			21			21										
	1	321	*311	21	121	*112		*301	*3021									
	2	31	*301		*112	*102	202	312	3121	112								
	3		*201			202	0011		*2121	*1111	*1101							
5/2	1			21	*301	312		*2211	202			111						
	2				*3021	3121	*2121	202	2221.2	*6222		*1111	*0002					
	3					112	*1111		*6222	*3212.002	3002	3112	5012	5022				
	4						*1101			3002	3321		*5202	*5101	*32			
7/2	2							111	*1111	3112		*2111	3002			3001		
	3								*0002	5012	*5202	3002	1102	*1312		*5101	*5300.1	
	4									5022	*5101		*1312	*5411	331	5311	5421.1	0321
	5										*32			331	2111.1		*3320.1	*1120.1
9/2	3											3001	*5101	5311		*3111,1	331	
	4												*5300.1	5421.1	*3320.1	331	3420.1 (41)²	*6320.2
	5													0321	*1120.1		*6320.2	*2120.[illegible]2

TABLE XIII

$W^2\left(l_1 J_1 l_2 J_2;\ \frac{3}{2}\ 2\right)$

J_2	l_2 \ J_1, l_1	1/2: 1	1/2: 2	3/2: 0	3/2: 1	3/2: 2	3/2: 3	5/2: 1	5/2: 2
1/2	1				311	301	201	201	2211
	2			201	322	3021	102	*212	*202
3/2	0		201			201			201
	1	311	322		112	102	*202	3021	322
	2	301	3021	201	102		*102	*3121	*3001
	3	201	102		*202	*102	*0121	2121	1222
5/2	1	201	*212		3021	*3121	2121	1221	*102
	2	2211	*202	201	322	*3001	1222	*102	*1102
	3	021	*102		*5221	*0021	1322	1021	2122,02
	4		*111			0111	1112		*2012
7/2	2			201	*102	1021	*3022	*102	1322
	3				*2111	3111	*2012	0111	1012
	4					3001	*2002		*3302
	5						*3001		
9/2	3							1101	*2002
	4								*2301.1
	5								

J_2	l_2 \ J_1, l_1	5/2: 3	5/2: 4	7/2: 2	7/2: 3	7/2: 4	7/2: 5	9/2: 3	9/2: 4	9/2: 5
1/2	1	021								
	2	*102	*111							
3/2	0			201						
	1	*5221		*102	*2111					
	2	*0021	0111	1021	3111	3001				
	3	1322	1112	*3022	*2012	*2002	*3001			
5/2	1	1021		*102	0111			1101		
	2	2122.02	*2012	1322	1012	*3302		*2002	*2301.1	
	3	4022,2	*4012	*4322	*5112,02	*5302,0002	4101	4102	4301.1	2101
	4	*4012	*4322.1	4212	5322	5212.1	431	*4312	*4211	*231
7/2	2	*4322	4212	2102	*4012	4102		*4002,1	4101	
	3	*5112.02	5322	*4012	*3222	3212	*4211	5212.1	5211	*3211.1
	4	*5302.0002	5212,1	4102	[illegible]	3322.12	*402	*5022.1	*5321.1 (71)^2	*1020.1
	5	4101	431		*4211	*402	*2221.11	4221.1	4021,2	1221.21
9/2	3	4102	*4312	*4002,1	5212.1	*5022.1	4221.1	4201.1	*402	2220.1
	4	4301.1	*4211	4101	5211	*5321.1(71)^2	4021.2	*402	*4303.2	2000.2
	5	2101	*231		*3211.1	*1020.1	1221.21	2220.1	2000.2	1200.21

TABLE XIV

$W^2\left(l_1J_1l_2J_2;\ \frac{3}{2}\,3\right)$

J_2	l_2 \ J_1	1/2		3/2				5/2				7/2				9/2		
	l_1	1	2	0	1	2	3	1	2	3	4	2	3	4	5	3	4	5
1/2	1								1201	2211	2001	2001	4101	4111				
	2							111	0011	2111	2101	*3011	*4001	*4211	*301			
3/2	0						2001			2001			2001			2001		
	1					202	0011		5221	4211	*4001	2111		*0201		*4211	*4210,1	
	2				202	102	0121	112	0021	*4121	*4111	*3111	*2101	*2111	311	4101	4120,1	012
	3			2001	0011	0121	1112	*1111	*1322	*3312	*3102,1	2012	3112	3102,1	2101	*3112	*3111	*1111
5/2	1		111			112	*1111		0121	*3111	3301	2011	*4111	4101		*3111.1	311	
	2	1201	0011		5221	0021	*1322	0121	*1022	*3122.002	3112	*3312	*4102	4312 (31)^2	*3111	3102.1	3321.02	*5121.1
	3	2211	2111	2001	4211	*4121	*3312	*3111	*3122.002	*3112	3102,1	3212	2112	2302,1	*3101	*3112.1	*3311.1(23)^2	*3112.1
	4	2001	2101		*4001	*4111	*3102.1	3301	*3112	3102,1	3422.1	*3202	*2302.1	*2202,1	*3301,01	3302	3211.1	3311.11
7/2	2	2001	*3011		2111	*3111	2012	2011	*3312	3212	*3202	*4112.1	3102.1	*3112.1	4111,1	3102.1	*3101	3101,1
	3	4101	*4001	2001		*2101	3112	*4111	*4102	2112	*2302.1	3102.1	4202.1	*4012	3211.1	*2202.1	*2001.1	4201.1
	4	4111	*4211		*0201	*2111	3102,1	4101	4312 (31)^2	2302.1	*2202.1	*3112.1	*4012	*4312.002	3011.11	2012	2321.1(31)^2	4221.11
	5		*301			311	2101		*3111	*3101	*3301.01	4111.1	3211.1	3011.11	4212.21	*3211.1	*3020.21	*3222.21
9/2	3			2001	*4211	4101	*3112	*3111.1	3102.1	*3112.1	3302	3102.1	*2202.1	2012	*3211.1	*3212.11	3011 11	*3211.11
	4				*4210.1	4120,1	*3111	311	3321.02	*3311.1(23)^2	3211.1	*3101	*2001.1	2321.1(31)^2	*3020.21	3011.11	3320.23	*3420.2
	5					012	*1111		*5121.1	*3112.1	3311.11	3101.1	4201.1	4221.11	*3222.21	*3211.11	*3420.2	*1222.2

TABLE XV

$$W^2\left(l_1J_1l_2J_2;\ \frac{3}{2}\ 4\right)$$

J_2	l_2 \ J_1	1/2		3/2				5/2				7/2				9/2		
	l_1	1	2	0	1	2	3	1	2	3	4	2	3	4	5	3	4	5
1/2	1												42	4301	22	22	2310.1	121
	2											301	41	4210,1	311	*212	*2221	*012
3/2	0										22			22			22	
	1									4001	41		0201	232	*2221	4210.1	*432	*5220.1
	2								0111	4111	4201.1	3001	2111	*2221.1	311	*4120.1	*4223.1	*0120.1
	3							1101	1112	3102.1	3202.1	*2002	*3102.1	*3222.1	*2121.01	3111	3220,1	1120.11
5/2	1						1101			3001	*33		4201.1	*4120.1	2221.1	3211.1	*3121	0221.1
	2					0111	1112		5212	3212	*3102.1	3102.1	*4012.1	*4122.0002	3220,1	*3222,1	*3121,1	1020.1
	3				4001	4111	3102.1	3001	3212	*3202.1	*3102.1	*3102,1	*2002	2122	3221.11	3212	3121,101	3021.11
	4			22	41	4201.1	3202.1	*33	*3102.1	*3102.1	*3402.11	3212	2322	2202,01	3300.11	*3312	*3201.11	*3300.11
7/2	2		301			3001	*2002		3102.1	*3102.1	3212	4012,1	*3102.1	3212	*4110,1	*3122,11	3221.11	*3120,11
	3	42	41		0201	2111	*3102.1	4201.1	*4012.1	*2002	2322	*3102,1	4202	4302,2	*3201,11	3202.01	2311.11	*4211.101
	4	4301	4210.1	22	232	*2221.1	*3222.1	*4120.1	*4122,0002	2122	2202.01	3212	4302.2	4402,01	*3300,11	*2302.01	*2431.1	*4310.1
	5	22	311		*2221	311	*2121,01	2221.1	3220,1	3221,11	3300.11	*4110.1	*3201.11	*3300.11	*4201.21	3201.1	3311.2	3211,2
9/2	3	22	*212		4210.1	*4120.1	3111	3211.1	*3222.1	3212	*3312	*3122.11	3202,01	*2302.01	3201.1	3211.01	*3300.11	3200.1
	4	2310.1	*2221	22	*432	*4223.1	3220.1	*3121	*3121.1	3121,101	*3201.11	3221.11	2311.11	*2431,1	3311.2	*3300,11	*3413.2	3310.2 (23)^2
	5	121	*012		*5220.1	*0120.1	1120.11	0221.1	1020.1	3021.11	*3300.11	*3120.11	*4211.101	*4310.1	3211,2	3200.1	3310.2 (23)^2	3210.2

TABLE XVI

$$Z^2\left(l_1 J_1 l_2 J_2; \frac{1}{2}\, 0\right)$$

J_2 \ J_1	l_2 \ l_1	1/2: 0	1/2: 1	3/2: 1	3/2: 2	5/2: 2	5/2: 3	7/2: 3	7/2: 4	9/2: 4	9/2: 5
1/2	0	1									
	1		1								
3/2	1			*2							
	2				*2						
5/2	2					11					
	3						11				
7/2	3							*3			
	4								*3		
9/2	4									101	
	5										101

TABLE XVII

$$Z^2\left(l_1 J_1 l_2 J_2; \frac{1}{2}\, 1\right)$$

J_2 \ J_1	l_2 \ l_1	1/2: 0	1/2: 1	3/2: 1	3/2: 2	5/2: 2	5/2: 3	7/2: 3	7/2: 4	9/2: 4	9/2: 5
1/2	0		1	2							
	1	*1			2						
3/2	1	*2			*201	*221					
	2		*2	201			*221				
5/2	2			221			1211	3201			
	3				221	*1211			3201		
7/2	3					*3201			*3101	*311	
	4						*3201	3101			*311
9/2	4							311			1110.1
	5								311	*1110.1	

TABLE XVIII

$$Z^2\left(l_1 J_1 l_2 J_2; \frac{1}{2}\, 2\right)$$

J_2 \ J_1	l_2 \ l_1	1/2: 0	1/2: 1	3/2: 1	3/2: 2	5/2: 2	5/2: 3	7/2: 3	7/2: 4	9/2: 4	9/2: 5
1/2	0				2	11					
	1			*2			11				
3/2	1		*2	*2			*2101	*3201			
	2	2			*2	2101			*3201		
5/2	2	11			2101	4101			3001	2021	
	3		11	*2101			4101	*3001			2021
7/2	3			*3201			*3001	*3121			*3121.1
	4				*3201	3001			*3121	3121.1	
9/2	4					2021			3121.1	4120.1	
	5						2021	*3121.1			4120.1

TABLE XIX

$Z^2\left(l_1 J_1 l_2 J_2; \frac{1}{2}\, 3\right)$

J_2	l_2 \ l_1 (J_1)	0 (1/2)	1 (1/2)	1 (3/2)	2 (3/2)	2 (5/2)	3 (5/2)	3 (7/2)	4 (7/2)	4 (9/2)	5 (9/2)
1/2	0						11	3			
	1					*11			3		
3/2	1				*221	*311			*31	*311	
	2			221			*311	31			*311
5/2	2		11	311			401	3			2010.1
	3	*11			311	*401			3	*2010.1	
7/2	3	*3			*31	*3			*3100.1	*3110.1	
	4		*3	31			*3	3100.1			*3110.1
9/2	4			311			*2010.1	3110.1			4110.11
	5				311	*2010.1			3110.1	*4110.11	

TABLE XX

$Z^2\left(l_1 J_1 l_2 J_2; \frac{1}{2}\, 4\right)$

J_2	l_2 \ l_1 (J_1)	0 (1/2)	1 (1/2)	1 (3/2)	2 (3/2)	2 (5/2)	3 (5/2)	3 (7/2)	4 (7/2)	4 (9/2)	5 (9/2)
1/2	0								3	101	
	1							*3			101
3/2	1						*3201	*3011			*3010.1
	2					3201			*3011	3010.1	
5/2	2				3201	2201			3211.1	4211.1	
	3			*3201			2201	*3211.1			4211.1
7/2	3		*3	*3011			*3211.1	*3401.1			*3411.11
	4	3			*3011	3211.1			*3401.1	3411.11	
9/2	4	101			3010.1	4211.1			3411.11	2410.11	
	5		101	*3010.1			4211.1	*3411.11			2410.11

TABLE XXI

$Z^2\left(l_1J_1l_2J_2;\frac{1}{2}5\right)$

J_2 \ J_1	l_2 \ l_1	$^1/_2$: 0	$^1/_2$: 1	$^3/_2$: 1	$^3/_2$: 2	$^5/_2$: 2	$^5/_2$: 3	$^7/_2$: 3	$^7/_2$: 4	$^9/_2$: 4	$^9/_2$: 5
$^1/_2$	0										101
	1									*101	
$^3/_2$	1								*$3\underline{1}1$	*$2\underline{1}1$	
	2							$3\underline{1}1$			*$2\underline{1}1$
$^5/_2$	2						$202\underline{1}$	$301\underline{1}$			$4010.0\underline{1}$
	3					*$202\underline{1}$			$301\underline{1}$	*$4010.0\underline{1}$	
$^7/_2$	3				*$3\underline{1}1$	*$301\underline{1}$			*$312\underline{1}.0\underline{1}$	*$3110.0\underline{1}$	
	4			$3\underline{1}1$			*$301\underline{1}$	$312\underline{1}.0\underline{1}$			*$3110.0\underline{1}$
$^9/_2$	4		101	$2\underline{1}1$			$4010.0\underline{1}$	$3110.0\underline{1}$			$2110.0\underline{1}$
	5	*101			$2\underline{1}1$	*$4010.0\underline{1}$			$3110.0\underline{1}$	*$2110.0\underline{1}$	

TABLE XXII

$Z^2(l_1J_1l_2J_2;\ 10)$

J_2 \ J_1	l_2 \ l_1	0: 1	1: 0	1: 1	1: 2	2: 1	2: 2	2: 3	3: 2	3: 3	3: 4	4: 3	4: 4	4: 5
0	1	*e												
1	0		01											
	1			01										
	2				01									
2	1					*001								
	2						*001							
	3							*001						
3	2								0001					
	3									0001				
	4										0001			
4	3											*02		
	4												*02	
	5													*02

TABLE XXIII

$Z^2(l_1J_1l_2J_2,\ 11)$

J_1		0	1			2			3			4		
J_2	l_2 \ l_1	1	0	1	2	1	2	3	2	3	4	3	4	5
0	1		*e		*1									
1	0	e		01		001								
	1		*01		11		12							
	2	1		*11		101		031						
2	1		*001		*101		*11		*1111					
	2			*12		11		*e		*3				
	3				*031		e		*0011		*2111			
3	2					1111		0011		e		0401		
	3						3		*e		21		221	
	4							2111		*21		2101		0111
4	3								*0401		*2101		*21	
	4									*221		21		*011
	5										*0111		011	

TABLE XXIV

$Z^2(l_1J_1l_2J_2;\ 12)$

J_1		0	1			2			3			4		
J_2	l_2 \ l_1	1	0	1	2	1	2	3	2	3	4	3	4	5
0	1					*1		*01						
1	0				01		001		0001					
	1			*11		*12		01		11				
	2		01		11		*101		1001		1301			
2	1	*1		*12		*1001		*0101		*01		*0401		
	2		001		*101		*1021		2011		*0111		*0121	
	3	*01		01		*0101		*5001		1		*1101		*1131
3	2		0001		1001		2011		4101		0001		0111	
	3			11		*01		1		2101		*212		011
	4				1301		*0111		0001		2141		*2211	
4	3					*0401		*1101		*212		*2021.1		*0011.1
	4						*0121		0111		*2211		*2201.102	
	5							*1131		011		*0011.1		*4001.1

TABLE XXV

$Z^2 (l_1 J_1 l_2 J_2;\ 13)$

J_2 \ J_1		0	1			2			3			4		
	l_2 \ l_1	1	0	1	2	1	2	3	2	3	4	3	4	5
0	1								*01		*2			
1	0							001		0001		02		
	1						*01		*11		12		111	
	2					031		301		*1001		1		101
2	1				*031		*11		*121		*111		*12	
	2			01		11		*2		*1101		112		*111
	3		*001		*301		2		*411		*001		01	
3	2	01		11		121		411		1101		1100,1		1110,1
	3		*0001		1001		1101		*1101		1001		1111,1	
	4	2		*12		111		001		*1001		1200,1		1210,1
4	3		*02		*1		*112		*1100,1		*1200,1		*1400,1	
	4			*111		12		*01		*1111,1		1400,1		*1410,1
	5				*101		111		*1110,1		*1210,1		1410,1	

TABLE XXVI

$Z^2 (l_1 J_1 l_2 J_2;\ 14)$

J_2 \ J_1		0	1			2			3			4		
	l_2 \ l_1	1	0	1	2	1	2	3	2	3	4	3	4	5
0	1											*2		*001
1	0										0001		02	
	1									*12		*111		11
	2								1301		1021		*12	
2	1							*2111		*111		*1011.1		*1001.1
	2						3011		1021		*1121		*1311,1	
	3					*2111		*1011		001		*0311.1		*4301,1
3	2				1301		1021		1001,1		1121,1		1501.1	
	3			*12		*111		001		1001.1		*1310,1		1300,1
	4		0001		1021		*1121		1121.1		1601,1		*1601.1	
4	3	*2		*111		*1011,1		*0311,1		*1310.1		*1401.11		*1411.11
	4		02		*12		*1311,1		1501.1		*1601,1		*1801.11	
	5	*001		11		*1001.1		*4301,1		1300,1		*1411.11		*1401.11

TABLE XXVII

$Z^2(l_1J_1l_2J_2;\ 15)$

J_2 \ J_1		0	1			2			3			4		
	l_2 \ l_1	1	0	1	2	1	2	3	2	3	4	3	4	5
0	1													
1	0													02
	1												*11	
	2											101		1
2	1										*0111		*02	
	2									012		012		*211
	3								*1131		*1011		11	
3	2							1131		012		0111.01		4101.01
	3						*012		*012		202		2100.01	
	4					0111		1011		*202		2231.01		0201.01
4	3				*101		*012		*0111.01		*2231.01		*2410.01	
	4			11		02		*11		*2100.01		2410.01		*0400.01
	5		*02		*1		211		*4101.01		*0201.01		0400.01	

TABLE XXVIII

$Z^2\left(l_1J_1l_2J_2;\frac{3}{2}\,0\right)$

J_2 \ J_1		1/2		3/2				5/2				7/2				9/2		
	l_2 \ l_1	1	2	0	1	2	3	1	2	3	4	2	3	4	5	3	4	5
1/2	1	*1																
	2		*1															
3/2	0			2														
	1				2													
	2					2												
	3						2											
5/2	1							*11										
	2								*11									
	3									*11								
	4										*11							
7/2	2											3						
	3												3					
	4													3				
	5														3			
9/2	3															*101		
	4																*101	
	5																	*101

TABLE XXIX

$Z^2\left(l_1 J_1 l_2 J_2; \frac{3}{2}\, 1\right)$

J_2	J_1 →	1/2		3/2				5/2				7/2				9/2		
	l_2 \ l_1	1	2	0	1	2	3	1	2	3	4	2	3	4	5	3	4	5
1/2	1		*1	*1		*1												
	2	1			*101		*121											
3/2	0	1			2			11										
	1		101	*2		602			1221									
	2	1			*602		222	112		422								
	3		121			*222			1221		4101							
5/2	1			*11		*112			*132			*411						
	2				*1221		*1221	132		*8222			*4302					
	3					*422			8222		*1302	*1212		*1222				
	4						*4101			1302			*1002		*101			
7/2	2							411		1212			3202			1211		
	3								4302		1002	*3202		7212			1210.1	
	4									1222			*7212		32	1201		4201
	5										101			*32			1200.1	
9/2	3											*1211		*1201			*121	
	4												*1210.1		*1200,1	121		*8210.2
	5													*4201			8210.2	

TABLE XXX

$Z^2\left(l_1J_1l_2J_2;\ \frac{3}{2}\,2\right)$

J_1		1/2		3/2				5/2				7/2				9/2		
J_2	l_2 \ l_1	1	2	0	1	2	3	1	2	3	4	2	3	4	5	3	4	5
1/2	1				*101		*121	*121		*211								
	2			1		1			*1101		*2201							
3/2	0		1			2			11			3						
	1	*101			*602		222	*1221		8121			3311					
	2		1	2					*1122		4202	5002		3212				
	3	*121			222		602	*1221		*4321			5111		3121			
5/2	1	*121			*1221		*1221	*2121		*1421			*4211			*1211		
	2		*1101	11		*1122			*2123		*1403	4403		*8013			*1022,1	
	3	*211			8121		*4321	*1421		*0121,2			1111,02		*1121	*0111		*1131
	4		*2201			4202			*1403		*0143	1103		1133			*0122,1	
7/2	2			3		5002			4403		1103	5123		3113			1122	
	3				3311		5111	*4211		1111,02			5321		3311	*1321,1		8321,1
	4					3212			*8013		1133	3113		5103,02			*1112,2(71)^{3}	
	5						3121			*1121			3311		5301	*1311,1		*4311,1
9/2	3							*1211		*0111			*1321,1		*1311,1	*0320,1		*1320,1
	4								*1022,1		*0122,1	1122		*1112,2(71)^{3}			*0142,3	
	5									*1131			8321,1		*4311,1	*1320,1		*2340,1

TABLE XXXI

$Z^2\left(l_1 J_1 l_2 J_2; \frac{3}{2}\, 3\right)$

J_2	l_2 \ J_1:	1/2		3/2				5/2	
	l_1:	1	2	0	1	2	3	1	2
1/2	1								*1101
	2							121	
3/2	0						2		
	1					*222			*8121
	2				222		602	422	
	3			*2		*602			4321
5/2	1		*121			*422			*2521
	2	1001			8121		*4321	2521	
	3		*211	*11		112			1021.002
	4	2201			*1201		*1201	1001	
7/2	2	2201			3311		5111	2311	
	3		21	*3		52			2101
	4	2011			*7101		3101	2101	
	5		211			*321			2111
9/2	3			*101		*121			*1111.1
	4				*1100,1		*1100,1	11	
	5					*421			8111.1

J_2	l_2 \ J_1:	5/2		7/2				9/2		
	l_1:	3	4	2	3	4	5	3	4	5
1/2	1		*2201	*2201		*2011				
	2	211			*21		*211			
3/2	0	11			3			101		
	1		1201	*3311		7101			1100,1	
	2	*112			*52		321	121		421
	3		1201	*5111		*3101			1100.1	
5/2	1		*1001	*2311		*2101			*11	
	2	*1021,002			*2101		*2111	1111.1		*8111.1
	3		*0301	*3211		*3001			0000,2(23)^2	
	4	0301			*3001		*3011	0011.1		1031.1
7/2	2	3211			5201.1		3211,1	3211.1		2231.1
	3		3001	*5201.1		3211,1			3210,2	
	4	3001			*3211.1		5201,1	3201.1		2401.1
	5		3011	*3211.1		*5201,1			3200.2	
9/2	3		*0011.1	*3211.1		*3201.1			*0210.21	
	4	*0000,2(23)^2			*3210.2		*3200,2	0210,21		*1610,21
	5		*1031.1	*2231,1		*2401.1			1610,21	

TABLE XXXII

$Z^2\left(l_1 J_1 l_2 J_2; \frac{3}{2}\, 4\right)$

J_2	l_2 \ J_1	1/2		3/2				5/2				7/2				9/2		
	l_1	1	2	0	1	2	3	1	2	3	4	2	3	4	5	3	4	5
1/2	1												*21		*211	*211		*111
	2											2201		2011			*2010.1	
3/2	0										11			3			101	
	1									*1201			*7101		3111	*1100.1		8100,1
	2								4202		1122	3212		*5002			*1012,2	
	3							*4101		*1201			3101		5111	*1100.1		*4100.1
5/2	1						*4101			*1301			*2001.1		*2011,1	*1001.1		*2001,1
	2					4202			8203		*1323	2213.1		*2203.1002			*1212,2	
	3				*1201		*1201	*1301		0201			3301.1		*3511,1	*0501,1		*1301.1
	4			11		1122			*1323		*0603	3133.1		3703.1			*0712,2	
7/2	2		2201			3212			2213.1		3133.1	3203.1		5413.1			3412,21	
	3	*21			*7101		3101	*2001,1		3301.1			*3001.1		5011.1	*3011,11		2011,112
	4		2011	3		*5002			*2203.1002		3703.1	5413.1		3403.1			*3442,21	
	5	*211			3111		5111	*2011,1		*3511.1			5011.1		3001,1	*3021,11		*2021.11
9/2	3	*211			*1100.1		*1100,1	*1001,1		*0501.1			*3011.11		*3021,11	*0010,11		*1030,11
	4		*2010.1	101		*1012,2			*1212.2		*0712,2	3412,21		*3442.21			*0412,31	
	5	*111			8100,1		*4100.1	*2001,1		*1301,1			2011,112		*2021.11	*1030,11		*6010.11

TABLE XXXIII

$Z^2\left(l_1J_1l_2J_2;\ \frac{3}{2}\,5\right)$

J_1		1/2		3/2				5/2				7/2				9/2		
J_2	l_2 \ l_1	1	2	0	1	2	3	1	2	3	4	2	3	4	5	3	4	5
1/2	1																*2010.1	
	2															211		111
3/2	0														3			101
	1													*3111			*8100.1	
	2												321		52	421		*121
	3											*3121		*5111			4100.1	
5/2	1										*101			*411			*1100.11	
	2									1032			8112		*41	1111.01		*2111.01
	3								*1032		*0312	*1222		1012,002			0000.1102	
	4							101		0312			*1032		*1	0031.01		1011.01
7/2	2						3121			1222			3222 01		5210,01	1211.01		4211.01
	3					*321			*8112		1032	*3222.01		5212,01			*1210.11	
	4				3111		5111	411		*1012.002			*5212,01		3200,01	1221,01		
	5			*3		*52			41		1	*5210.01		*3200,01			1220.11	
9/2	3		*211			*421			*1111.01		*0031.01	*1211.01		*1221,01			*0230.11	
	4	2010.1			8100,1		*4100.1	1100,11		*0000.1102			1210,11		*1220,11	0230,11		*1410.11
	5		*111	*101		121			2111.01		*1011,01	*4211,01					1410,11	

TABLE XXXIV

$Z_{\bar{1}}^{2}\left(l_1J_1l_2J_2;\frac{1}{2}0\right)$

J_2	J_1 / l_2 \ l_1	1/2		3/2		5/2	
		0	1	1	2	2	3
1/2	0						
	1		*1				
3/2	1			2			
	2				2		
5/2	2					*11	
	3						*11

TABLE XXXV

$Z_{\bar{1}}^{2}\left(l_1J_1l_2J_2;\frac{1}{2}1\right)$

J_2	J_1 / l_2 \ l_1	1/2		3/2		5/2	
		0	1	1	2	2	3
1/2	0						
	1		*1	*e*	*01		
3/2	1		*e*	001	01$\underline{1}$	03$\underline{1}$	
	2		*01	01$\underline{1}$	021	*00$\underline{1}$	50$\underline{1}$
5/2	2			03$\underline{1}$	*00$\underline{1}$	*10$\underline{1}$1	*40$\underline{11}$
	3				50$\underline{1}$	*40$\underline{11}$	*101$\underline{1}$

TABLE XXXVI

$Z_{\bar{1}}^{2}\left(l_1J_1l_2J_2;\frac{1}{2}2\right)$

J_2	J_1 / l_2 \ l_1	1/2		3/2		5/2	
		0	1	1	2	2	3
1/2	0						
	1			*e*	*01	1	*2
3/2	1		*e*	*e*	01	0001	300$\underline{1}$
	2		*01	01	**e*	010$\underline{1}$	310$\underline{1}$
5/2	2		1	0001	010$\underline{1}$	210$\underline{1}$	*110$\underline{1}$
	3		*2	300$\underline{1}$	310$\underline{1}$	*110$\underline{1}$	030$\underline{1}$

TABLE XXXVII

$Z_{\bar{1}}^{2}\left(l_1J_1l_2J_2;\frac{1}{2}3\right)$

J_2	J_1 / l_2 \ l_1	1/2		3/2		5/2	
		0	1	1	2	2	3
1/2	0						
	1					1	*2
3/2	1				21$\underline{1}$	30$\underline{1}$	001
	2			21$\underline{1}$	*40$\underline{1}$	31$\underline{1}$	*01$\underline{1}$
5/2	2		1	30$\underline{1}$	31$\underline{1}$	22$\underline{1}$	10$\underline{1}$
	3		*2	001	*01$\underline{1}$	10$\underline{1}$	001

TABLE XXXVIII

$$Z^2_{\bar{1}}\left(l_1J_1l_2J_2; \frac{1}{2}\,4\right)$$

J_2	l_2 \ l_1	J_1 = 1/2: 0	1/2: 1	3/2: 1	3/2: 2	5/2: 2	5/2: 3
1/2	0						
	1						
3/2	1						0301
	2					5001	*0021
5/2	2				5001	4001	1021
	3			0301	*0021	1021	*0001

TABLE XXXIX

$$X^2\begin{pmatrix} a & b & 1/2 \\ d & e & 1/2 \\ 1 & 1 & 1 \end{pmatrix}$$

a	b	d	e	X^2	a	b	d	e	X^2
1/2	1	1/2	0	13	3/2	1	5/2	2	*421
3/2	1	1/2	0	33	3/2	2	5/2	2	43
1/2	0	1/2	1	13	5/2	3	3/2	2	1211
3/2	1	1/2	1	42	7/2	3	5/2	2	3301
3/2	2	1/2	1	*43	3/2	2	5/2	3	331
1/2	0	3/2	1	*33	5/2	2	5/2	3	1211
1/2	1	3/2	1	42	7/2	3	5/2	3	53
3/2	2	3/2	1	131	7/2	4	5/2	3	*5201
5/2	2	3/2	1	421	5/2	2	7/2	3	*3301
1/2	1	3/2	2	43	5/2	3	7/2	3	53
3/2	1	3/2	2	131	7/2	4	7/2	3	1401
5/2	2	3/2	2	43	5/2	3	7/2	4	5201
5/2	3	3/2	2	*331	7/2	3	7/2	4	1401

TABLE XL

$$X^2\begin{pmatrix} a & b & 1/2 \\ d & e & 1/2 \\ 2 & 2 & 1 \end{pmatrix}$$

a	b	d	e	X^2	a	b	d	e	X^2
3/2	2	1/2	0	302	3/2	1	5/2	[illegible]	*4421
5/2	2	1/2	0	122	3/2	2	5/2	2	421
3/2	1	1/2	1	411	5/2	3	5/2	2	1121
3/2	2	1/2	1	*422	7/2	3	5/2	2	4121
5/2	2	1/2	1	242	7/2	4	5/2	2	4522
5/2	3	1/2	1	*141	1/2	1	5/2	3	141
1/2	1	3/2	1	411	3/2	1	5/2	3	3411
3/2	2	3/2	1	122	3/2	2	5/2	3	2221
5/2	2	3/2	1	4421	5/2	2	5/2	3	1121
5/2	3	3/2	1	3411	7/2	3	5/2	3	521
7/2	3	3/2	1	2211	7/2	4	5/2	3	*5511
1/2	0	3/2	2	302	3/2	1	7/2	3	*2211
1/2	1	3/2	2	422	3/2	2	7/2	3	3021
3/2	1	3/2	2	122	5/2	2	7/2	3	*4121
5/2	2	3/2	2	421	5/2	3	7/2	3	521
5/2	3	3/2	2	*2221	7/2	4	7/2	3	1311
7/2	3	3/2	2	3021	3/2	2	7/2	4	322
7/2	4	3/2	2	*322	5/2	2	7/2	4	4522
1/2	0	5/2	2	*122	5/2	3	7/2	4	5511
1/2	1	5/2	2	242	7/2	3	7/2	4	1311

TABLE XLI

$$X^2\begin{pmatrix} a & b & 1/2 \\ d & e & 1/2 \\ 3 & 3 & 1 \end{pmatrix}$$

a	b	d	e	X²	a	b	d	e	X²
5/2	3	1/2	0	0202	7/2	4	5/2	2	6501
7/2	3	1/2	0	4002	1/2	0	5/2	3	0202
5/2	2	1/2	1	0401	1/2	1	5/2	3	0402
5/2	3	1/2	1	*0402	3/2	1	5/2	3	5432
7/2	3	1/2	1	6222	3/2	2	5/2	3	5112
7/2	4	1/2	1	*6101	5/2	2	5/2	3	1112
3/2	2	3/2	1	1211	7/2	3	5/2	3	5201
5/2	2	3/2	1	3411	7/2	4	5/2	3	*5502.1
5/2	3	3/2	1	5432	1/2	0	7/2	3	*4002
7/2	3	3/2	1	5102	1/2	1	7/2	3	6222
7/2	4	3/2	1	5201	3/2	1	7/2	3	*5102
3/2	1	3/2	2	1211	3/2	2	7/2	3	5002
5/2	2	3/2	2	4201	5/2	2	7/2	3	*6202
5/2	3	3/2	2	*5112	5/2	3	7/2	3	5201
7/2	3	3/2	2	5002	7/2	4	7/2	3	1302
7/2	4	3/2	2	*5301	1/2	1	7/2	4	6101
1/2	1	5/2	2	0401	3/2	1	7/2	4	5201
3/2	1	5/2	2	*3411	3/2	2	7/2	4	5301
3/2	2	5/2	2	4201	5/2	2	7/2	4	6501
5/2	3	5/2	2	1112	5/2	3	7/2	4	5502.1
7/2	3	5/2	2	6202	7/2	3	7/2	4	1302

TABLE XLII

$$X^2\begin{pmatrix} a & b & 1/2 \\ d & e & 1/2 \\ 4 & 4 & 1 \end{pmatrix}$$

a	b	d	e	X²	a	b	d	e	X²
7/2	4	1/2	0	451	5/2	2	5/2	3	1311
7/2	3	1/2	1	641	7/2	3	5/2	3	54
7/2	4	1/2	1	*64	7/2	4	5/2	3	*5411
5/2	3	3/2	1	5311	1/2	1	7/2	3	641
7/2	3	3/2	1	5401	3/2	1	7/2	3	*5401
7/2	4	3/2	1	5411	3/2	2	7/2	3	5121
5/2	2	3/2	2	44	5/2	2	7/2	3	*6421.1
5/2	3	3/2	2	*5411	5/2	3	7/2	3	54
7/2	3	3/2	2	5121	7/2	4	7/2	3	1501
7/2	4	3/2	2	*5520.1	1/2	0	7/2	4	451
3/2	2	5/2	2	44	1/2	1	7/2	4	64
5/2	3	5/2	2	1311	3/2	1	7/2	4	5411
7/2	3	5/2	2	6421.1	3/2	2	7/2	4	5520.1
7/2	4	5/2	2	6422	5/2	2	7/2	4	6422
3/2	1	5/2	3	5311	5/2	3	7/2	4	5411
3/2	2	5/2	3	5411	7/2	3	7/2	4	1501

TABLE XLIII

$$X^2\begin{pmatrix} a & b & c \\ a & e & f \\ k & k & 1 \end{pmatrix}$$

a	b	c	a	e	f	X^2 k=2	X^2 k=4
1	1	1	1	1	2	*223	
1	2	1	1	2	2	2141	
3/2	1	1/2	3/2	1	3/2	612	
3/2	1	3/2	3/2	1	5/2	*612	
3/2	2	1/2	3/2	2	3/2	6031	
3/2	2	3/2	3/2	2	5/2	612	
2	1	1	2	1	2	2141	
2	2	1	2	2	2	2121	*1421
5/2	1	3/2	5/2	1	5/2	322	
5/2	2	1/2	5/2	2	3/2	3131	4301
5/2	2	3/2	5/2	2	5/2	0122	*5312
3	2	1	3	2	2	0242	2412,1

TABLE XLIV

$$X^2\begin{pmatrix} a & b & c \\ a & e & c \\ k & k & 1 \end{pmatrix}$$

a	b	c	a	e	c	X^2 k=2	X^2 k=4
1	1	2	1	3	2	222	
3/2	1	3/2	3/2	3	3/2	322	
3/2	1	5/2	3/2	3	5/2	1321	
2	1	1	2	3	1	2311	3301
2	1	2	2	3	2	1221	*4411
5/2	1	3/2	5/2	3	3/2	2221	5301
5/2	1	5/2	5/2	3	5/2	1123	*2603
3	1	2	3	3	2	0222	4502,1

TABLE XLV

$[\Delta(abc)]^{-2}$; a, b, c — Integral numbers

a b	c=0	1	2	3	4	5	6
1 1	01	31	111				
2 1		111	311	0111			
2 2	001	311	1211	5011	1211		
3 2		0111	5011	2311	4211	1111.1	
3 3	0001	4101	2311	5121	1221.1	4301.1	2101.11
4 2			1211	4211	2211.1	5210.1	0210.11
4 3		2201	4211	1221.1	5211.1	2211.11	4201.11
4 4	02	421	2211.1	5211.1	1221.11	4202.11	2212.11
5 3			1111.1	4301.1	2211.11	5111.11	0321.11
5 4		0210.1	5210.1	2211.11	4202.11	1222.11	7211.11
5 5	0000.1	3110.1	1310.11	5111.11	1222.11	6302.11	4112.111
6 3				2101.11	4201.11	0321.11	6021.11
6 4			0210.11	4201.11	2212.11	7211.11	2221.111
6 5		1100.11	3011.11	0321.11	7211.11	4112.111	6401.111
6 6	0000.01	3101.01	1211.11	6021.11	2221.111	6401.111	4202.1111
7 4				0211.11	7210.11	3211.111	5211.111
7 5			0111.11	5111.11	3211.111	7310.111	2311.1111
7 6		0111,01	6111.01	3111.111	5211.111	2311.1111	7311.1111
7 7	011	5111	3111.011	6211.011	2211.1111	5211.1111	3312.1111
8 4					1210.111	5210.111	3201.1111
8 5				3001.111	5210.111	1410.1111	6120.1111
8 6			2011.011	5301.011	3201.1111	6120.1111	1421.1111
8 7		3110.001	5211.001	2211.0111	6211.0111	3212.1111	5211.2111
8 8	0000.001	5200.001	3310.0011	6121.0011	2222.0111	5302.1111	3102.2111.1

TABLE XLVI

$[\Delta(abc)]^{-2}$; *a, b* — Semi-integral numbers

a, b	c=0	1	2	3	4	5	6
1/2 1/2	1	11					
3/2 1/2		21	201				
3/2 3/2	2	211	221	2011			
5/2 1/2			111	1101			
5/2 3/2		211	2111	3111	3201		
5/2 5/2	11	1111	4111	4211	2221	2201.1	
7/2 1/2				3001	32		
7/2 3/2			3011	3301	3211	3110.1	
7/2 5/2		3101	3211	3221	3211.1	3211.1	3200.11
7/2 7/2	3	3201	3311	3121.1	3221.1	3301.11	3102.11
9/2 1/2					121	1010.1	
9/2 3/2				3111	3210.1	2310.1	2010.11
9/2 5/2			2211	2211.1	4211.1	4210.11	1211.11
9/2 7/2		321	3111.1	3311.1	3211.11	3112.11	3311.11
9/2 9/2	101	1210.1	4310.1	4111.11	2212.11	2312.11	6112.11
11/2 1/2						2100.1	2100.01
11/2 3/2					2210.1	2110.11	2101.11
11/2 5/2				3201.1	3210.11	2211.11	2121.11
11/2 7/2			3210.1	3201.11	3202.11	3221.11	4221.11
11/2 9/2		2110.1	2210.11	3211.11	3212.11	5212.11	5211.111
11/2 11/2	21	2100.11	2111.11	2221.11	5221.11	5202.111	5302.111

	c=7	8	9	10	11
7/2 7/2	3210.11				
9/2 5/2	1111.11				
9/2 7/2	4211.11	4010.111			
9/2 9/2	6210.111	2410.111	2010.1111		
11/2 3/2	2111.01				
11/2 5/2	4111.11	4101.011			
11/2 7/2	4111.111	4201.111	4200.0111		
11/2 9/2	4311.111	4210.1111	3210.1111	3111.0111	
11/2 11/2	5310.1111	3320.1111	3121.1111	3101.2111	3101.0111.1

TABLE XLVII

$(l_1 0 l_2 0 \mid L0)^2$

l_1 l_2	L=0	2	4	6	8	10	12
0 0	e						
1 1	*01	11					
2 2	001	*1001	1211				
3 3	*0001	2101	*1201.1	2121.1			
4 4	02	*2221.1	1401.11	*2210.1	1212.11		
5 5	*0000.1	1120.11	*1200.11	*4110,101	*1012.1101	2402,1111	
6 6	0000,01	*1001.11	2201.111	*4020,1011	1021.1101	*2402,0111,1	2201.2111,1
0 2		e					
0 4			e				
0 6				e			
1 3		*0101	2001				
1 5			*0010.1	1100.1			
2 4		1001	*2011.1	0010.1			
2 6			0210.11	*1011.1	2011.01		
3 5		*1121.1	2211.11	*0101.1	3001.11		
4 6		0020.11	*2010.11	2001.101	*3201.1101	1112.0111	

l_1 l_2	L=1	3	5	7	9	11	
0 1	e						
0 3		e					
0 5			e				
1 2	*101	011					
1 4		*22	021				
1 6			*1100.01	0001.01			
2 3	0211	*211	1111				
2 5		1110.1	*1110.01	0201.11			
3 4	*2101	-1000.1	*2011.01	0121.11			
3 6		*2120.11	0101.01	*3201.111	2101,011		
4 5	0110,1	*2010.11	1000.01	*3121.111	1202.111		
5 6	*1200.11	0102,11	*4110.011	2221.1111	*2111.111	1201.1111	

TABLE XLVIII

$(l_1 - 1 l_2 1 \mid L0)^2$

l_1 l_2	$L=0$	1	2	3	4	5	6
1 1	$0\underline{1}$	$*\underline{1}$	$\underline{11}$				
2 2	$*00\underline{1}$	$\underline{1}0\underline{1}$	$\underline{1}00\underline{1}$	$*10\underline{1}$	$30\underline{11}$		
3 3	$000\underline{1}$	$*\underline{2}00\underline{1}$	$*\underline{2}10\underline{1}$	$\underline{11}$	$\underline{1}00\underline{1.1}$	$*\underline{21}2\underline{1}$	$\underline{2}12\underline{1.1}$
1 2		$\underline{1}1\underline{1}$	$*\underline{1}$	$00\underline{1}$			
1 3			$100\underline{1}$	$*\underline{1}$	$\underline{1}10\underline{1}$		
2 3		$*30\underline{11}$	$000\underline{1}$	$\underline{111}$	$*\underline{1}01\underline{1}$	$0\underline{1}1\underline{1}$	

TABLE XLIX

Factorials

$1! = e$	$7! = 4211$	$13! = \bar{0}521.11$	$19! = \bar{6}832.1111$
$2! = 1$	$8! = 7211$	$14! = \bar{1}522.11$	$20! = \bar{8}842.1111$
$3! = 11$	$9! = 7411$	$15! = \bar{1}632.11$	$21! = \bar{8}943.1111$
$4! = 31$	$10! = 8421$	$16! = \bar{5}632.11$	$22! = \bar{9}943.2111$
$5! = 311$	$11! = 8421.1$	$17! = \bar{5}632.111$	$23! = \bar{9}943.2111.1$
$6! = 421$	$12! = \bar{0}521.1$	$18! = \bar{6}832.111$	$24! = \bar{\bar{2}}043.2111.1$

TABLE L

$\left(j_1 m_1 \frac{1}{2} m_2 \middle| jm\right)$

$j=$	$m_2 = \frac{1}{2}$	$m_2 = -\frac{1}{2}$
$j_1 + \frac{1}{2}$	$\sqrt{\dfrac{j_1 + m + \frac{1}{2}}{2j_1+1}}$	$\sqrt{\dfrac{j_1 - m + \frac{1}{2}}{2j_1+1}}$
$j_1 - \frac{1}{2}$	$-\sqrt{\dfrac{j_1 - m + \frac{1}{2}}{2j_1+1}}$	$\sqrt{\dfrac{j_1 + m + \frac{1}{2}}{2j_1+1}}$

TABLE LI

$(j_1 m_1 1 m_2 \mid jm)$

$j=$	$m_2 = 1$	$m_2 = 0$	$m_2 = -1$
$j_1 + 1$	$\sqrt{\dfrac{(j_1+m)(j_1+m+1)}{(2j_1+1)(2j_1+2)}}$	$\sqrt{\dfrac{(j_1-m+1)(j_1+m+1)}{(2j_1+1)(j_1+1)}}$	$\sqrt{\dfrac{(j_1-m)(j_1-m+1)}{(2j_1+1)(2j_1+2)}}$
j_1	$-\sqrt{\dfrac{(j_1+m)(j_1-m+1)}{2j_1(j_1+1)}}$	$\dfrac{m}{\sqrt{j_1(j_1+1)}}$	$\sqrt{\dfrac{(j_1-m)(j_1+m+1)}{2j_1(j_1+1)}}$
$j_1 - 1$	$\sqrt{\dfrac{(j_1-m)(j_1-m+1)}{2j_1(2j_1+1)}}$	$-\sqrt{\dfrac{(j_1-m)(j_1+m)}{j_1(2j_1+1)}}$	$\sqrt{\dfrac{(j_1+m+1)(j_1+m)}{2j_1(2j_1+1)}}$

TABLE LII

$$\left(j_1 m_1 \tfrac{3}{2} m_2 \middle| jm\right)$$

$j=$	$m_2=\frac{3}{2}$	$m_2=\frac{1}{2}$
$j_1+\frac{3}{2}$	$\sqrt{\frac{\left(j_1+m-\frac{1}{2}\right)\left(j_1+m+\frac{1}{2}\right)\left(j_1+m+\frac{3}{2}\right)}{(2j_1+1)(2j_1+2)(2j_1+3)}}$	$\sqrt{\frac{3\left(j_1+m+\frac{1}{2}\right)\left(j_1+m+\frac{3}{2}\right)\left(j_1-m+\frac{3}{2}\right)}{(2j_1+1)(2j_1+2)(2j_1+3)}}$
$j_1+\frac{1}{2}$	$-\sqrt{\frac{3\left(j_1+m-\frac{1}{2}\right)\left(j_1+m+\frac{1}{2}\right)\left(j_1-m+\frac{3}{2}\right)}{2j_1(2j_1+1)(2j_1+3)}}$	$-\left(j_1-3m+\frac{3}{2}\right)\sqrt{\frac{j_1+m+\frac{1}{2}}{2j_1(2j_1+1)(2j_1+3)}}$
$j_1-\frac{1}{2}$	$\sqrt{\frac{3\left(j_1+m-\frac{1}{2}\right)\left(j_1-m+\frac{1}{2}\right)\left(j_1-m+\frac{3}{2}\right)}{(2j_1-1)(2j_1+1)(2j_1+2)}}$	$-\left(j_1+3m-\frac{1}{2}\right)\sqrt{\frac{j_1-m+\frac{1}{2}}{(2j_1-1)(2j_1+1)(2j_1+2)}}$
$j_1-\frac{3}{2}$	$-\sqrt{\frac{\left(j_1-m-\frac{1}{2}\right)\left(j_1-m+\frac{1}{2}\right)\left(j_1-m+\frac{3}{2}\right)}{2j_1(2j_1-1)(2j_1+1)}}$	$\sqrt{\frac{3\left(j_1+m-\frac{1}{2}\right)\left(j_1-m-\frac{1}{2}\right)\left(j_1-m+\frac{1}{2}\right)}{2j_1(2j_1-1)(2j_1+1)}}$

$j=$	$m_2=-\frac{1}{2}$	$m_2=-\frac{3}{2}$
$j_1+\frac{3}{2}$	$\sqrt{\frac{3\left(j_1+m+\frac{3}{2}\right)\left(j_1-m+\frac{1}{2}\right)\left(j_1-m+\frac{3}{2}\right)}{(2j_1+1)(2j_1+2)(2j_1+3)}}$	$\sqrt{\frac{\left(j_1-m-\frac{1}{2}\right)\left(j_1-m+\frac{1}{2}\right)\left(j_1-m+\frac{3}{2}\right)}{(2j_1+1)(2j_1+2)(2j_1+3)}}$
$j_1+\frac{1}{2}$	$\left(j_1+3m+\frac{3}{2}\right)\sqrt{\frac{j_1-m+\frac{1}{2}}{2j_1(2j_1+1)(2j_1+3)}}$	$\sqrt{\frac{3\left(j_1+m+\frac{3}{2}\right)\left(j_1-m-\frac{1}{2}\right)\left(j_1-m+\frac{1}{2}\right)}{2j_1(2j_1+1)(2j_1+3)}}$
$j_1-\frac{1}{2}$	$-\left(j_1-3m-\frac{1}{2}\right)\sqrt{\frac{j_1+m+\frac{1}{2}}{(2j_1-1)(2j_1+1)(2j_1+2)}}$	$\sqrt{\frac{3\left(j_1+m+\frac{1}{2}\right)\left(j_1+m+\frac{3}{2}\right)\left(j_1-m-\frac{1}{2}\right)}{(2j_1-1)(2j_1+1)(2j_1+2)}}$
$j_1-\frac{3}{2}$	$-\sqrt{\frac{3\left(j_1+m-\frac{1}{2}\right)\left(j_1+m+\frac{1}{2}\right)\left(j_1-m-\frac{1}{2}\right)}{2j_1(2j_1-1)(2j_1+1)}}$	$\sqrt{\frac{\left(j_1+m-\frac{1}{2}\right)\left(j_1+m+\frac{1}{2}\right)\left(j_1+m+\frac{3}{2}\right)}{2j_1(2j_1-1)(2j_1+1)}}$

TABLE LIII

$(j_1 m_1 2 m_2 \mid jm)$

$j=$	$m_2=2$	$m_2=1$	$m_2=0$
j_1+2	$\sqrt{\frac{(j_1+m-1)(j_1+m)(j_1+m+1)(j_1+m+2)}{(2j_1+1)(2j_1+2)(2j_1+3)(2j_1+4)}}$	$\sqrt{\frac{(j_1-m+2)(j_1+m+2)(j_1+m+1)(j_1+m)}{(2j_1+1)(j_1+1)(2j_1+3)(j_1+2)}}$	$\sqrt{\frac{3(j_1-m+2)(j_1-m+1)}{(2j_1+1)(2j_1+2)(2j_1+3)(j_1+2)}}\times$ $\times\sqrt{(j_1+m+2)(j_1+m+1)}$
j_1+1	$-\sqrt{\frac{(j_1+m-1)(j_1+m)(j_1+m+1)(j_1-m+2)}{2j_1(j_1+1)(j_1+2)(2j_1+1)}}$	$-(j_1-2m+2)\sqrt{\frac{(j_1+m+1)(j_1+m)}{2j_1(2j_1+1)(j_1+1)(j_1+2)}}$	$m\sqrt{\frac{3(j_1-m+1)(j_1+m+1)}{j_1(2j_1+1)(j_1+1)(j_1+2)}}$
j_1	$\sqrt{\frac{3(j_1+m-1)(j_1+m)(j_1-m+1)(j_1-m+2)}{(2j_1-1)2j_1(j_1+1)(2j_1+3)}}$	$(1-2m)\sqrt{\frac{3(j_1-m+1)(j_1+m)}{(2j_1-1)j_1(2j_1+2)(2j_1+3)}}$	$\frac{3m^2-j_1(j_1+1)}{\sqrt{(2j_1-1)j_1(j_1+1)(2j_1+3)}}$
j_1-1	$\sqrt{\frac{(j_1+m-1)(j_1-m)(j_1-m+1)(j_1-m+2)}{2(j_1-1)j_1(j_1+1)(2j_1+1)}}$	$(j_1+2m-1)\sqrt{\frac{(j_1-m+1)(j_1-m)}{(j_1-1)j_1(2j_1+1)(2j_1+2)}}$	$-m\sqrt{\frac{3(j_1-m)(j_1+m)}{(j_1-1)j_1(2j_1+1)(j_1+1)}}$
j_1-2	$\sqrt{\frac{(j_1-m-1)(j_1-m)(j_1-m+1)(j_1-m+2)}{(2j_1-2)(2j_1-1)2j_1(2j_1+1)}}$	$-\sqrt{\frac{(j_1-m+1)(j_1-m)(j_1-m-1)(j_1+m-1)}{(j_1-1)(2j_1-1)j_1(2j_1+1)}}$	$\sqrt{\frac{3(j_1-m)(j_1-m-1)(j_1+m)(j_1+m-1)}{(2j_1-2)(2j_1-1)j_1(2j_1+1)}}$

$j=$	$m_2=-1$	$m_2=-2$
j_1+2	$\sqrt{\frac{(j_1-m+2)(j_1-m+1)(j_1-m)(j_1+m+2)}{(2j_1+1)(j_1+1)(2j_1+3)(j_1+2)}}$	$\sqrt{\frac{(j_1-m-1)(j_1-m)(j_1-m+1)(j_1-m+2)}{(2j_1+1)(2j_1+2)(2j_1+3)(2j_1+4)}}$
j_1+1	$(j_1+2m+2)\sqrt{\frac{(j_1-m+1)(j_1-m)}{j_1(2j_1+1)(2j_1+2)(j_1+2)}}$	$\sqrt{\frac{(j_1-m-1)(j_1-m)(j_1-m+1)(j_1+m+2)}{j_1(2j_1+1)(j_1+1)(2j_1+4)}}$
j_1	$(2m+1)\sqrt{\frac{3(j_1-m)(j_1+m+1)}{(2j_1-1)j_1(2j_1+2)(2j_1+3)}}$	$\sqrt{\frac{3(j_1-m-1)(j_1-m)(j_1+m+1)(j_1+m+2)}{(2j_1-1)j_1(2j_1+2)(2j_1+3)}}$
j_1-1	$-(j_1-2m-1)\sqrt{\frac{(j_1+m+1)(j_1+m)}{(j_1-1)j_1(2j_1+1)(2j_1+2)}}$	$\sqrt{\frac{(j_1-m-1)(j_1+m)(j_1+m+1)(j_1+m+2)}{(j_1-1)j_1(2j_1+1)(2j_1+2)}}$
j_1-2	$-\sqrt{\frac{(j_1-m-1)(j_1+m+1)(j_1+m)(j_1+m-1)}{(j_1-1)(2j_1-1)j_1(2j_1+1)}}$	$\sqrt{\frac{(j_1+m-1)(j_1+m)(j_1+m+1)(j_1+m+2)}{(2j_1-2)(2j_1-1)2j_1(2j_1+1)}}$

TABLE LIV

$$W\left(l_1J_1l_2J_2;\ \frac{1}{2};\ L\right)$$

	$l_1=J_1+\frac{1}{2}$	$l_1=J_1-\frac{1}{2}$
$l_2=J_2+\frac{1}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{(J_1+J_2+L+2)(J_1+J_2-L+1)}{(2J_1+1)(2J_1+2)(2J_2+1)(2J_2+2)}\right]^{\frac{1}{2}}$	$(-1)^{J_1+J_2-L}\left[\frac{(L-J_1+J_2+1)(L+J_1-J_2)}{(2J_1)(2J_1+1)(2J_2+1)(2J_2+2)}\right]^{\frac{1}{2}}$
$l_2=J_2-\frac{1}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{(L+J_1-J_2+1)(L-J_1+J_2)}{(2J_1+1)(2J_1+2)(2J_2)(2J_2+1)}\right]^{\frac{1}{2}}$	$(-1)^{J_1+J_2-L-1}\left[\frac{(J_1+J_2+L+1)(J_1+J_2-L)}{(2J_1)(2J_1+1)(2J_2)(2J_2+1)}\right]^{\frac{1}{2}}$

TABLE LV

$$W(l_1J_1l_2J_2;\ 1,\ L)$$

	$l_2=J_2+1$
$l_1=J_1+1$	$(-1)^{J_1+J_2-L}\left[\frac{(L+J_1+J_2+3)(L+J_1+J_2+2)(-L+J_1+J_2+2)(-L+J_1+J_2+1)}{4(2J_1+3)(J_1+1)(2J_1+1)(2J_2+3)(J_2+1)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_1=J_1$	$(-1)^{J_1+J_2-L}\left[\frac{(L+J_1+J_2+2)(-L+J_1+J_2+1)(L-J_1+J_2+1)(L+J_1-J_2)}{4J_1(2J_1+1)(J_1+1)(2J_2+1)(J_2+1)(2J_2+3)}\right]^{\frac{1}{2}}$
$l_1=J_1-1$	$(-1)^{J_1+J_2-L}\left[\frac{(L+J_1-J_2)(L+J_1-J_2-1)(L-J_1+J_2+2)(L-J_1+J_2+1)}{4(2J_1+1)(2J_1-1)(J_1)(J_2+1)(2J_2+1)(2J_2+3)}\right]^{\frac{1}{2}}$
	$l_2=J_2$
$l_1=J_1+1$	$(-1)^{J_1+J_2-L}\left[\frac{(L+J_1+J_2+2)(L+J_1-J_2+1)(J_1+J_2-L+1)(L-J_1+J_2)}{4(2J_1+1)(J_1+1)(2J_1+3)(J_2)(J_2+1)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_1=J_1$	$(-1)^{J_1+J_2-L-1}\left[\frac{J_1(J_1+1)+J_2(J_2+1)-L(L+1)}{[4J_1(J_1+1)(2J_1+1)(J_2)(J_2+1)(2J_2+1)]^{\frac{1}{2}}}\right]$
$l_1=J_1-1$	$(-1)^{J_1+J_2-L-1}\left[\frac{(L+J_1+J_2+1)(-L+J_1+J_2)(L+J_1-J_2)(L-J_1+J_2+1)}{4(2J_1+1)(J_1)(2J_1-1)(J_2)(2J_2+1)(J_2+1)}\right]^{\frac{1}{2}}$
	$l_2=J_2-1$
$l_1=J_1+1$	$(-1)^{J_1+J_2-L}\left[\frac{(L-J_1+J_2)(L-J_1+J_2-1)(L+J_1-J_2+2)(L+J_1-J_2+1)}{4(2J_1+1)(J_1+1)(2J_1+3)(2J_2-1)(J_2)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_1=J_1$	$(-1)^{J_1+J_2-L-1}\left[\frac{(L+J_1+J_2+1)(L+J_1-J_2+1)(L+J_2-J_1)(J_1+J_2-L)}{4J_1(2J_1+1)(J_1+1)(J_2)(2J_2+1)(2J_2-1)}\right]^{\frac{1}{2}}$
$l_1=J_1-1$	$(-1)^{J_1+J_2-L}\left[\frac{(L+J_1+J_2+1)(L+J_1+J_2)(-L+J_1+J_2)(-L+J_1+J_2-1)}{4(2J_1+1)(J_1)(2J_1-1)(2J_2+1)(J_2)(2J_2-1)}\right]^{\frac{1}{2}}$

TABLE LVI

$$W\left(l_1 J_1 l_2 J_2, \frac{3}{2} L\right)$$

	$l_1 = J_1 + {}^3/_2$
$l_2 = J_2 + \frac{3}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{(L+J_1+J_2+4)(L+J_1+J_2+3)(L+J_1+J_2+2)(-L+J_1+J_2+3)(-L+J_1+J_2+2)(-L+J_1+J_2+1)}{(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_2 = J_2 + \frac{1}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{3(L+J_1+J_2+3)(L+J_1+J_2+2)(L+J_1-J_2+1)(L-J_1+J_2)(-L+J_1+J_2+2)(-L+J_1+J_2+1)}{(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)}\right]^{\frac{1}{2}}$
$l_2 = J_2 - \frac{1}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{3(L+J_1+J_2+2)(L+J_1-J_2+2)(L+J_1-J_2+1)(L-J_1+J_2)(L-J_1+J_2-1)(J_1+J_2-L+1)}{(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)}\right]^{\frac{1}{2}}$
$l_2 = J_2 - \frac{3}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{(L-J_1+J_2)(L-J_1+J_2-1)(L-J_1+J_2-2)(L+J_1-J_2+3)(L+J_1-J_2+2)(L+J_1-J_2+1)}{(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)}\right]^{\frac{1}{2}}$

	$l_1 = J_1 + \frac{1}{2}$
$l_2 = J_2 + \frac{3}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{3(L+J_1+J_2+3)(L+J_1+J_2+2)(-L+J_1+J_2+2)(-L+J_1+J_2+1)(L-J_1+J_2+1)(L+J_1-J_2)}{(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_2 = J_2 + \frac{1}{2}$	$(-1)^{J_1+J_2-L-1}\,\frac{[(L+J_1+J_2+2)(-L+J_1+J_2+1)]^{\frac{1}{2}}\,[(L+J_1+J_2+3)(-L+J_1+J_2)-2(L-J_1+J_2)(L+J_1-J_2)]}{[(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)]^{1/2}}$
$l_2 = J_2 - \frac{1}{2}$	$(-1)^{J_1+J_2-L-1}\,\frac{[(L+J_1-J_2+1)(L-J_1+J_2)]^{\frac{1}{2}}\,[2(L+J_1+J_2+2)(-L+J_1+J_2)-(L-J_1+J_2-1)(L+J_1-J_2)]}{[(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)]^{1/2}}$
$l_2 = J_2 - \frac{3}{2}$	$(-1)^{J_1+J_2-L-1}\left[\frac{3(L+J_1+J_2+1)(L-J_1+J_2)(L-J_1+J_2-1)(L+J_1-J_2+2)(L+J_1-J_2+1)(-L+J_1+J_2)}{(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)}\right]^{\frac{1}{2}}$

TABLE LVI (contd.)

	$l_1 = J_1 - \frac{1}{2}$
$l_2 = J_2 + \frac{3}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{3(L+J_1+J_2+2)(-L+J_1+J_2+1)(L-J_1+J_2+2)(L-J_1+J_2+1)(L+J_1-J_2)(L+J_1-J_2-1)}{(2J_1+2)(2J_1+1)(2J_1)(2J_1-1)(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_2 = J_2 + \frac{1}{2}$	$(-1)^{J_1+J_2-L}\frac{[(L-J_1+J_2+1)(L+J_1-J_2)]^{\frac{1}{2}}[(L-J_1+J_2)(L+J_1-J_2-1)-2(L+J_1+J_2+2)(-L+J_1+J_2)]}{[(2J_1+2)(2J_1+1)(2J_1)(2J_1-1)(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)]^{1/2}}$
$l_2 = J_2 - \frac{1}{2}$	$(-1)^{J_1+J_2-L}\frac{[(L+J_1+J_2+1)(-L+J_1+J_2)]^{\frac{1}{2}}[(L+J_1+J_2+2)(-L+J_1+J_2-1)-2(L+J_1-J_2)(L-J_1+J_2)]}{[(2J_1+2)(2J_1+1)(2J_1)(2J_1-1)(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)]^{1/2}}$
$l_2 = J_2 - \frac{3}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{3(L+J_1+J_2+1)(L+J_1+J_2)(L-J_1+J_2)(L+J_1-J_2+1)(-L+J_1+J_2)(-L+J_1+J_2-1)}{(2J_1+2)(2J_1+1)(2J_1)(2J_1-1)(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)}\right]^{\frac{1}{2}}$

	$l_1 = J_1 - \frac{3}{2}$
$l_2 = J_2 + \frac{3}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{(L+J_1-J_2)(L+J_1-J_2-1)(L+J_1-J_2-2)(L-J_1+J_2+3)(L-J_1+J_2+2)(L-J_1+J_2+1)}{(2J_1+1)(2J_1)(2J_1-1)(2J_1-2)(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_2 = J_2 + \frac{1}{2}$	$(-1)^{J_1+J_2-L-1}\left[\frac{3(L+J_1+J_2+1)(-L+J_1+J_2)(L+J_1-J_2)(L+J_1-J_2-1)(L-J_1+J_2+2)(L-J_1+J_2+1)}{(2J_1+1)(2J_1)(2J_1-1)(2J_1-2)(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)}\right]^{\frac{1}{2}}$
$l_2 = J_2 - \frac{1}{2}$	$(-1)^{J_1+J_2-L}\left[\frac{3(L+J_1+J_2+1)(L+J_1+J_2)(-L+J_1+J_2)(-L+J_1+J_2-1)(L+J_1-J_2)(L-J_1+J_2+1)}{(2J_1+1)(2J_1)(2J_1-1)(2J_1-2)(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)}\right]^{\frac{1}{2}}$
$l_2 = J_2 - \frac{3}{2}$	$(-1)^{J_1+J_2-L-1}\left[\frac{(L+J_1+J_2+1)(L+J_1+J_2)(L+J_1+J_2-1)(-L+J_1+J_2)(-L+J_1+J_2-1)(-L+J_1+J_2-2)}{(2J_1+1)(2J_1)(2J_1-1)(2J_1-2)(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)}\right]^{\frac{1}{2}}$

TABLE LVII

$W(l_1J_1l_2J_2;\,2L)$

	$l_1=J_1+2$
$l_2=J_2+2$	$(-1)^{L-J_1-J_2}\left[\dfrac{(L+J_1+J_2+5)(L+J_1+J_2+4)(L+J_1+J_2+3)(L+J_1+J_2+2)\times(-L+J_1+J_2+4)(-L+J_1+J_2+3)(-L+J_1+J_2+2)(-L+J_1+J_2+1)}{(2J_1+5)(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)\cdot(2J_2+5)(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_2=J_2+1$	$(-1)^{L-J_1-J_2}\left[\dfrac{4(L+J_1+J_2+4)(L+J_1+J_2+3)(L+J_1+J_2+2)(L+J_1-J_2+1)\times(-L+J_1+J_2+3)(-L+J_1+J_2+2)(-L+J_1+J_2+1)(L-J_1+J_2)}{(2J_1+5)(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)}\right]^{\frac{1}{2}}$
$l_2=J_2$	$(-1)^{L-J_1-J_2}\left[\dfrac{6(L+J_1+J_2+3)(L+J_1+J_2+2)(L+J_1-J_2+2)(L+J_1-J_2+1)\times(-L+J_1+J_2+2)(-L+J_1+J_2+1)(L-J_1+J_2)(L-J_1+J_2-1)}{(2J_1+5)(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)\cdot(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)}\right]^{\frac{1}{2}}$
$l_2=J_2-1$	$(-1)^{L-J_1-J_2}\left[\dfrac{4(L+J_1+J_2+2)(L-J_1+J_2)(L-J_1+J_2-1)(L-J_1+J_2-2)(L+J_1-J_2+3)\times(L+J_1-J_2+2)(L+J_1-J_2+1)(-L+J_1+J_2+1)}{(2J_1+5)(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)\cdot(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)}\right]^{\frac{1}{2}}$
$l_2=J_2-2$	$(-1)^{L-J_1-J_2}\left[\dfrac{(L-J_1+J_2)(L-J_1+J_2-1)(L-J_1+J_2-2)(L-J_1+J_2-3)\times(L+J_1-J_2+4)(L+J_1-J_2+3)(L+J_1-J_2+2)(L+J_1-J_2+1)}{(2J_1+5)(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)\cdot(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)(2J_2-3)}\right]^{\frac{1}{2}}$

	$l_1=J_1-1$
$l_2=J_2+2$	$(-1)^{L-J_1-J_2}\left[\dfrac{4(L+J_1+J_2+2)(-L+J_1+J_2+1)(L-J_1+J_2+3)(L-J_1+J_2+2)\times(L-J_1+J_2+1)(L+J_1-J_2)(L+J_1-J_2-1)(L+J_1-J_2-2)}{(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)(2J_1+2)\cdot(2J_2+1)(2J_2+2)(2J_2+3)(2J_2+4)(2J_2+5)}\right]^{\frac{1}{2}}$
$l_2=J_2+1$	$(-1)^{L-J_1-J_2-1}\left[\dfrac{(L-J_1+J_2+2)(L-J_1+J_2+1)(L+J_1-J_2)(L+J_1-J_2-1)}{(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)(2J_1+2)\cdot(2J_2)(2J_2+1)(2J_2+2)(2J_2+3)(2J_2+4)}\right]^{\frac{1}{2}}\times$ $\times 4\cdot[(J_1-1)(J_1+J_2+2)-(L+J_2+2)(L-J_2-1)]$
$l_2=J_2$	$(-1)^{L-J_1-J_2}\left[\dfrac{6(L+J_1+J_2+1)(L+J_1-J_2)(-L+J_1+J_2)(L-J_1+J_2+1)}{(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)(2J_1+2)\cdot(2J_2-1)(2J_2)(2J_2+1)(2J_2+2)(2J_2+3)}\right]^{\frac{1}{2}}\times$ $\times 2\cdot[J_1^2-1-(L+J_2+1)(L-J_2)]$
$l_2=J_2-1$	$(-1)^{L-J_1-J_2-1}\left[\dfrac{(L+J_1+J_2+1)(L+J_1+J_2)(-L+J_1+J_2)(-L+J_1+J_2-1)}{(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)(2J_1+2)\cdot(2J_2-2)(2J_2-1)(2J_2)(2J_2+1)(2J_2+2)}\right]^{\frac{1}{2}}\times$ $\times 4\,[(J_1-1)(J_1-J_2+1)-(L+J_2)(L-J_2+1)]$
$l_2=J_2-2$	$(-1)^{L-J_1-J_2-1}\left[\dfrac{4(L+J_1+J_2+1)(L+J_1+J_2)(L+J_1+J_2-1)(L+J_1-J_2+1)\times(-L+J_1+J_2)(-L+J_1+J_2-1)(-L+J_1+J_2-2)(L-J_1+J_2)}{(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)(2J_1+2)\cdot(2J_2-3)(2J_2-2)(2J_2-1)(2J_2)(2J_2+1)}\right]^{\frac{1}{2}}$

TABLE LVII (contd.)

	$l_1=J_1-2$
$l_2=J_2+2$	$(-1)^{L-J_1-J_2}\left[\frac{(L+4-J_1+J_2)(L+3-J_1+J_2)(L+2-J_1+J_2)(L+1-J_1+J_2)\times(L+J_1-J_2)(L+J_1-J_2-1)(L+J_1-J_2-2)(L+J_1-J_2-3)}{(2J_1-3)(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)\cdot(2J_2+5)(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_2=J_2+1$	$(-1)^{L-J_1-J_2-1}\left[\frac{4(L+1+J_1+J_2)(L+3-J_1+J_2)(L+2-J_1+J_2)(L+1-J_1+J_2)\times(L+J_1-J_2)(L+J_1-J_2-1)(L+J_1-J_2-2)(-L+J_1+J_2)}{(2J_1-3)(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)\cdot(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)}\right]^{\frac{1}{2}}$
$l_2=J_2$	$(-1)^{L-J_1-J_2}\left[\frac{6(L+1+J_1+J_2)(L+J_1+J_2)(L+2-J_1+J_2)(L+1-J_1+J_2)\times(L+J_1-J_2)(L+J_1-J_2-1)(-L+J_1+J_2)(-L+J_1+J_2-1)}{(2J_1-3)(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)\cdot(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)}\right]^{\frac{1}{2}}$
$l_2=J_2-1$	$(-1)^{L-J_1-J_2}\left[\frac{4(L+1+J_1+J_2)(L+J_1+J_2)(L+J_1+J_2-1)(L+1-J_1+J_2)\times(L+J_1-J_2)(-L+J_1+J_2)(-L+J_1+J_2-1)(-L+J_1+J_2-2)}{(2J_1-3)(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)\cdot(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)}\right]^{\frac{1}{2}}$
$l_2=J_2-2$	$(-1)^{L-J_1-J_2}\left[\frac{(L+1+J_1+J_2)(L+J_1+J_2)(L+J_1+J_2-1)(L+J_1+J_2-2)\times(-L+J_1+J_2)(-L+J_1+J_2-1)(-L+J_1+J_2-2)(-L+J_1+J_2-3)}{(2J_1-3)(2J_1-2)(2J_1-1)(2J_1)(2J_1+1)\cdot(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)(2J_2-3)}\right]^{\frac{1}{2}}$

	$l_1=J_1+1$
$l_2=J_2+2$	$(-1)^{L-J_1-J_2}\left[\frac{4(L+J_1+J_2+4)(L+J_1+J_2+3)(L+J_1+J_2+2)(L-J_1+J_2+1)\times(L+J_1-J_2)(-L+J_1+J_2+3)(-L+J_1+J_2+2)(-L+J_1+J_2+1)}{(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)\cdot(2J_2+5)(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_2=J_2+1$	$(-1)^{L-J_1-J_2-1}\left[\frac{(L+J_1+J_2+3)(L+J_1+J_2+2)(-L+J_1+J_2+2)(-L+J_1+J_2+1)}{(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)\cdot(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)}\right]^{\frac{1}{2}}\times 4\cdot[(J_1+1)(J_1-J_2)-L(L+1)+J_2(J_2+2)]$
$l_2=J_2$	$(-1)^{L-J_1-J_2-1}\left[\frac{6(L+J_1+J_2+2)(L-J_1+J_2)(L+J_1-J_2+1)(-L+J_1+J_2+1)}{(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)\cdot(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)}\right]^{\frac{1}{2}}\times 2\cdot[J_1(J_1+2)+J_2(J_2+1)-L(L+1)]$
$l_2=J_2-1$	$(-1)^{L-J_1-J_2-1}\left[\frac{(L-J_1+J_2)(L-J_1+J_2-1)(L+J_1-J_2+2)(L+J_1-J_2+1)}{(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)\cdot(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)}\right]^{\frac{1}{2}}\times 4[J_1(J_1+2)+J_2(J_2+1)+J_1J_2-L(L+1)]$
$l_2=J_2-2$	$(-1)^{L-J_1-J_2-1}\left[\frac{4(L+J_1+J_2+1)(-L+J_1+J_2)(L-J_1+J_2)(L-J_1+J_2-1)\times(L-J_1+J_2-2)(L+J_1-J_2+3)(L+J_1-J_2+2)(L+J_1-J_2+1)}{(2J_1+4)(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)\cdot(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)(2J_2-3)}\right]^{\frac{1}{2}}$

TABLE LVII (contd.)

	$l_1 = J_1$
$l_2 = J_2 + 2$	$(-1)^{L-J_1-J_2}\left[\dfrac{6(L+J_1+J_2+3)(L+J_1+J_2+2)(L+2-J_1+J_2)(L+1-J_1+J_2)\times(L+J_1-J_2)(L+J_1-J_2-1)(-L+J_1+J_2+2)(-L+J_1+J_2+1)}{(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)(2J_1-1)\cdot(2J_2+5)(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)}\right]^{\frac{1}{2}}$
$l_2 = J_2 + 1$	$(-1)^{L-J_1-J_2-1}\left[\dfrac{6(L+J_1+J_2+2)(L+J_1-J_2)(L-J_1+J_2+1)(-L+J_1+J_2+1)}{(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)(2J_1-1)\cdot(2J_2+4)(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)}\right]^{\frac{1}{2}}\times$ $\times 2\left[J_1(J_1+1)+J_2(J_2+2)-L(L+1)\right]$
$l_2 = J_2$	$(-1)^{L-J_1-J_2}\left[\dfrac{1}{(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)(2J_1-1)\cdot(2J_2+3)(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)}\right]^{\frac{1}{2}}\times$ $\times 6\cdot\left[A(A+1)-\frac{4}{3}J_1(J_1+1)J_2(J_2+1)\right]$, где $A \equiv L(L+1)-J_1(J_1+1)-l_2(J_2+1)$
$l_2 = J_2 - 1$	$(-1)^{L-J_4-J_2}\left[\dfrac{6(L+J_1+J_2+1)(L-J_1+J_2)(L+J_1-J_2+1)(-L+J_1+J_2)}{(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)(2J_1-1)\cdot(2J_2+2)(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)}\right]^{\frac{1}{2}}\times$ $\times 2\cdot\left[J_1(J_1+1)-L(L+1)+J_2^2-1\right]$
$l_2 = J_2 - 2$	$(-1)^{L-J_1-J_2}\left[\dfrac{6(L+J_1+J_2+1)(L+J_1+J_2)(L-J_1+J_2)(L-J_1+J_2-1)\times(-L+J_1+J_2)(-L+J_1+J_2-1)(L+J_1-J_2+2)(L+J_1-J_2+1)}{(2J_1+3)(2J_1+2)(2J_1+1)(2J_1)(2J_1-1)\cdot(2J_2+1)(2J_2)(2J_2-1)(2J_2-2)(2J_2-3)}\right]^{\frac{1}{2}}$

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